A TREATISE

ON

HYDRODYNAMICS

With numerous Examples.

BY

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PREFACE.

THE second volume of this Treatise deals with the more advanced portions of Hydrodynamics, including the motion of viscous liquids to which the last four chapters have been devoted. It commences with a chapter on Harmonic Analysis, in which a variety of functions which frequently occur in physical investigations are considered. The most exhaustive work on this subject is the German Treatise on *Kugelfunctionen* by Heine, of which considerable use has been made, especially in the first twenty pages of this chapter. The remainder of the chapter which relates to Toroidal Functions, is taken from Mr Hicks' papers in the *Philosophical Transactions* for 1881 and 1884.

The notation $J_m(x)$ for an ordinary Bessel's Function of degree m is well established, and the second solution of Bessel's equation, which is not however so frequently required, may be conveniently denoted by $Y_m(x)$; but there is another class of functions also of considerable importance, which constitute the two solutions of the equation which is obtained by changing x into ιx in Bessel's equation. The notation for these functions does not appear to be so well established, many English writers employing the symbols $J_m(\iota x)$ and $Y_m(\iota x)$, whilst German writers often employ the symbol $K_m(\iota x)$ in the place of $Y_m(\iota x)$. But as it appears to me that the employment of an imaginary argument in the case of functions which may always be treated as real quantities, creates unnecessary complexity, I have ventured to introduce a new notation, and have denoted these functions by the symbols $I_m(x)$ and $K_m(x)$ respectively.

The portions of Chapter XIV. which relate to the vibrations of a circular vortex and to linked vortices, have been taken with slight modifications from a paper by Professor J. J. Thomson in the *Philosophical Transactions* for 1882, and from the Treatise on the *Motion of Vortex Rings* by the same author, to which the Adams' Prize was adjudged in 1882. The latter portion of this chapter has been derived from Mr Hicks' papers on vortex rings in the *Philosophical Transactions* for 1884 and 1885. It is however necessary to point out, that the period equation obtained by Mr Hicks for determining the fluted vibrations of a circular vortex, does not agree with that obtained by myself, and consequently there is an important difference in the results connected with the stability of the vortex. I am however indebted to Mr A. E. H. Love, for having examined and verified the analysis of §§ 326—340, and I therefore trust that the results which are put forward are the correct ones.

In the Chapter on Waves, I have made considerable use of Prof. Greenhill's Article on Waves in the *American Journal of Mathematics*, Vol. IX., which contains an exhaustive discussion of most of the principal problems of interest.

The Chapter on the Tides is confined exclusively to the dynamical theories which have been proposed as an explanation of tidal phenomena, and is principally derived from the investigations of the late Astronomer Royal and Professor G. H. Darwin. The reduction of tidal observations, together with a variety of questions relating to the practical portion of the subject, are very fully treated in Professor Darwin's Article on Tides in the *Encyclopaedia Britannica.*

Although nearly forty years have elapsed since the publication of Prof. Stokes' paper "On the Effects of the Internal Friction of Fluids on Pendulums," it is remarkable that very little progress has been made with respect to the solution of problems connected with the motion of solid bodies in a viscous liquid. The complete solutions for a sphere and a right circular cylinder moving in a

viscous liquid of unlimited extent under the action of given forces, have not yet been obtained; and no problem involving the motion of *two* solids appears to have ever been attempted; neither have any general equations analogous to Lagrange's equations been discovered, by means of which the motion of one or more solids in a viscous liquid may be obtained, without going through the troublesome process of calculating the components of the force and couple exerted by the liquid on each solid. The difficulties of the subject are undoubtedly great, but it is hoped that before the termination of the present century, substantial progress will be made.

I have in conclusion to express my obligations to Professor Greenhill for having read the proof sheets; to Mr A. E. H. Love for having examined the analysis of §§ 326—340, and for having read the proof sheets of the last four chapters; and to Professor J. J. Thomson and Professor G. H. Darwin for permission to make free use of their investigations on Vortex Rings and Laplace's Theory of the Tides respectively.

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,,	249	In the first table *read* ν for μ.
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CHAPTER XII.

ON SPHEROIDAL HARMONICS AND ALLIED FUNCTIONS.

242. It is shown in treatises on Spherical Harmonics, that every spherical harmonic of degree n, is expressible in a series of the form $\Sigma_{m=0}^{m=n} A_m P_n^m(\mu)\sin(m\phi + \alpha_m)$, where $\cos^{-1}\mu$ and ϕ are the co-latitude and longitude of a point on a sphere, and $P_n^m(\mu)$ is called an *associated function of the first kind of degree* n *and order* m. This function satisfies the equation

$$\frac{d}{d\mu}(1-\mu^2)\frac{d\psi}{d\mu} - \frac{m^2\psi}{1-\mu^2} + n(n+1)\psi = 0 \quad\text{.........(1).}$$

This differential equation being of the second order has two independent integrals. The first of these is $P_n^m(\mu)$, and is finite for all finite values of μ, and is infinite when $\mu = \infty$. The second integral, which will be denoted by $Q_n^m(\mu)$, is as we shall presently show, infinite when $\mu = \pm 1$, but is finite for all other values of μ, and vanishes when $\mu = \pm\infty$.

243. Laplace, to whom we are indebted for the invention of spherical harmonic analysis, principally devoted his attention to the attractions of spheres, and of bodies slightly differing therefrom; and it was therefore sufficient for him to consider the properties of the first solution upon the supposition that $\mu < 1$; but in dealing with the potentials of ovary ellipsoids, the function P_n^m is required both when $\mu < 1$ and $\mu > 1$; and the function Q_n^m is required when $\mu > 1$. We shall therefore consider these functions from their most general point of view, and shall denote the argument by μ when it is < 1, and by ν when it is > 1.

244. The function P_n^m may be very briefly dismissed.

It is shown in Ferrers' *Spherical Harmonics* that

$$P_n^m(\mu) = (1-\mu^2)^{\frac{1}{2}m} \frac{d^m P_n}{d\mu^m} \quad \mu < 1 \quad \text{.............(2)},$$

or

$$P_n^m(\nu) = (\nu^2-1)^{\frac{1}{2}m} \frac{d^m P_n}{d\nu^m} \quad \nu > 1 \quad \text{.............(3)},$$

where P_n is an ordinary zonal harmonic or Legendre's coefficient. The value of P_n can be expressed either in the form of a terminating series of powers of μ, or by means of the definite integral

$$P_n = \frac{1}{\pi}\int_0^\pi \{\mu + \sqrt{(\mu^2-1)}\cos\theta\}^n d\theta$$

$$= \frac{1}{\pi}\int_0^\pi \{\mu + \sqrt{(\mu^2-1)}\cos\theta\}^{-n-1} d\theta \quad \text{.........(4)}.$$

The expressions for P_n in terms of the series, or in terms of either of the definite integrals, hold good whether $\mu <$ or > 1.

An expression for P_n^m in the form of a definite integral may be found as follows. Let

$$V_m = \int_0^\pi \frac{\sin^{2m}\theta d\theta}{\{\mu + \sqrt{(\mu^2-1)}\cos\theta\}^{n+m+1}}.$$

Then

$$\frac{dV_m}{d\mu} = -\frac{n+m+1}{\sqrt{(\mu^2-1)}}\int_0^\pi \frac{\{\sqrt{(\mu^2-1)}+\mu\cos\theta\}\sin^{2m}\theta d\theta}{\{\mu+\sqrt{(\mu^2-1)}\cos\theta\}^{n+m+2}}$$

$$= -\frac{n+m+1}{\sqrt{(\mu^2-1)}}\int_0^\pi \frac{\cos\theta\sin^{2m}\theta d\theta}{\{\mu+\sqrt{(\mu^2-1)}\cos\theta\}^{n+m+1}} - (n+m+1)V_{m+1}.$$

Integrating by parts we obtain

$$\frac{dV_m}{d\mu} = \frac{(n+m+1)(n-m)}{2m+1}V_{m+1}.$$

Now $V_0 = \pi P_n$, therefore

$$\frac{d^m P_n}{d\mu^m} = \frac{(n-m+1)(n-m+2)\ldots(n+m)}{1\,.\,3\,.\,5\ldots(2m-1)\,\pi}V_m,$$

whence

$$P_n^m = \frac{(n+m)!\,(1-\mu^2)^{\frac{1}{2}m}}{\pi\,(n-m)!\,1\,.\,3\ldots(2m-1)}\int_0^\pi \frac{\sin^{2m}\theta d\theta}{\{\mu+\sqrt{(\mu^2-1)}\cos\theta\}^{n+m+1}} \quad \text{...(5)}.$$

If we transform the definite integral by putting

$$\cos\theta = \frac{\mu\cos\phi + \sqrt{(\mu^2-1)}}{\mu + \sqrt{(\mu^2-1)}\cos\phi},$$

we obtain

$$P_n^m = \frac{(n+m)!\,(1-\mu^2)^{\frac{1}{2}m}}{\pi(n-m)!\,1.3\ldots(2m-1)} \int_0^\pi \{\mu + \sqrt{(\mu^2-1)}\cos\phi\}^{n-m} \sin^{2m}\phi\, d\phi . \quad (6).$$

If $\mu > 1 = \nu$, we must change the factor $(1-\mu^2)^{\frac{1}{2}m}$ into $(\nu^2-1)^{\frac{1}{2}m}$ in (5) and (6).

245. We shall now consider the function Q_n^m.

Let us first suppose that $m = 0$; writing ν for μ, (1) becomes

$$\frac{d}{d\nu}(1-\nu^2)\frac{dQ_n}{d\nu} + n(n+1)Q_n = 0 \quad \ldots\ldots\ldots\ldots \quad (7).$$

If we endeavour to express Q_n in the form of a series of powers of ν^{-1}, it will be found that

$$Q_n = \frac{1}{\nu^{n+1}} \sum_{r=0}^{r=\infty} \frac{(2r+1)(2r+2)\ldots(2r+n)}{(2r+1)(2r+3)\ldots(2r+2n+1)} \frac{1}{\nu^{2r}} \quad \ldots\ldots (8).$$

This series is convergent if $\nu > 1$, but when $\nu \leqq 1$ it is divergent.

246. A series for Q_n in powers of ν could easily be obtained when $\nu < 1$, but it will not be required; we shall therefore proceed to find an expression for Q_n in the form of a definite integral.

Let
$$U = \frac{1}{2H}\log\frac{\nu + x + H}{\nu + x - H},$$

where
$$H^2 = 1 + 2\nu x + x^2 \text{ and } \nu > 1.$$

Then

$$(1-\nu^2)\frac{dU}{d\nu} = \frac{1}{H^2}\{1 + \nu x - x(1-\nu^2)U\},$$

$$\frac{d}{d\nu}(1-\nu^2)\frac{dU}{d\nu} = \frac{x}{H^2}\{x^2 - \nu x - 2 + (\nu^2 x + 2\nu x^2 + 2\nu + 3x)U\}.$$

Also

$$\frac{d}{dx}(xU) = \frac{1}{H^2}\{x + (1+\nu x)U\},$$

$$x\frac{d^2}{dx^2}(xU) = \frac{x}{H^2}\{2 + \nu x - x^2 - (\nu^2 x + 2\nu x^2 + 2\nu + 3x)U\};$$

therefore
$$\frac{d}{d\nu}(1-\nu^2)\frac{dU}{d\nu} + x\frac{d^2}{dx^2}(xU) = 0.$$

Hence if $U = \Sigma S_n x^n$, S_n satisfies the equation

$$\frac{d}{d\nu}(1-\nu^2)\frac{dS_n}{d\nu} + n(n+1)S_n = 0;$$

and since $\nu > 1$, S_n must be equal to $A_n Q_n(\nu)$ where A_n is some constant. Now if $a > b$,

$$\int_0^\infty \frac{d\theta}{a + b\cosh\theta} = \frac{1}{\sqrt{(a^2 - b^2)}} \log \frac{a + \sqrt{(a^2 - b^2)}}{b}.$$

Putting $a = \nu + x$, $b = \sqrt{(\nu^2 - 1)}$,

$$\int_0^\infty \frac{d\theta}{\nu + \sqrt{(\nu^2 - 1)}\cosh\theta + x} = \frac{1}{(1 + 2\nu x + x^2)^{\frac{1}{2}}} \log \frac{\nu + x + \sqrt{(1 + 2\nu x + x^2)}}{\sqrt{(\nu^2 - 1)}}$$

$$= \frac{1}{2H} \log \frac{\nu + x + H}{\nu + x - H}$$

$$= \Sigma A_n Q_n x^n.$$

Expanding the definite integral and equating the coefficients of x^n, we find that

$$(-1)^n A_n Q_n = \int_0^\infty \frac{d\theta}{\{\nu + \sqrt{(\nu^2 - 1)}\cosh\theta\}^{n+1}}.$$

If the left-hand side be expanded in powers of ν^{-1}, the coefficient of ν^{-n-1}—the first term in the expansion—is evidently equal to

$$\int_0^\infty \frac{d\theta}{(1 + \cosh\theta)^{n+1}} = \int_0^1 (1 - z)^n z^{-\frac{1}{2}} dz$$

$$= \frac{n!}{1 \,.\, 3 \ldots (2n + 1)};$$

comparing this with the series (8) for Q_n, we see that $A_n = (-1)^n$, whence

$$Q_n = \int_0^\infty \frac{d\theta}{\{\nu + \sqrt{(\nu^2 - 1)}\cosh\theta\}^{n+1}} \quad \ldots\ldots\ldots\ldots\ldots (9).$$

247. We can now establish the following equations, viz.

$$(n + 2) Q_{n+2} - (2n + 3) Q_{n+1} + (n + 1) Q_n = 0 \quad \ldots\ldots (10),$$

$$\frac{\nu^2 - 1}{n + 1} \frac{dQ_n}{d\nu} = Q_{n+1} - \nu Q_n \quad \ldots\ldots\ldots\ldots\ldots\ldots (11),$$

$$\frac{\nu^2 - 1}{n} \frac{dQ_n}{d\nu} = \nu Q_n - Q_{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12).$$

We obtain from (9)

$$\frac{dQ_n}{d\nu} = -\frac{(n + 1)}{\sqrt{(\nu^2 - 1)}} \int_0^\infty \frac{\{\sqrt{(\nu^2 - 1)} + \nu\cosh\theta\}\, d\theta}{\{\nu + \sqrt{(\nu^2 - 1)}\cosh\theta\}^{n+2}},$$

therefore

$$\frac{\nu^2 - 1}{n + 1} \frac{dQ_n}{d\nu} = -\int_0^\infty \frac{\{\nu^2 - 1 + \nu\sqrt{(\nu^2 - 1)}\cosh\theta\}\, d\theta}{\{\nu + \sqrt{(\nu^2 - 1)}\cosh\theta\}^{n+2}} \ldots (13),$$

$$= -\nu Q_n + Q_{n+1},$$

which proves (11). Again from (13) we obtain

$$\frac{\nu^2-1}{n+1}\frac{dQ_n}{d\nu} = -\int_0^\infty \frac{\sqrt{(\nu^2-1)}\cosh\theta\, d\theta}{\{\nu+\sqrt{(\nu^2-1)}\cosh\theta\}^{n+1}}$$

$$+ (\nu^2-1)\int_0^\infty \frac{\sinh^2\theta\, d\theta}{\{\nu+\sqrt{(\nu^2-1)}\cosh\theta\}^{n+2}}.$$

Integrating the last term by parts, the right-hand side

$$= -\frac{n}{n+1}\cdot\int_0^\infty \frac{\sqrt{(\nu^2-1)}\cosh\theta\, d\theta}{\{\nu+\sqrt{(\nu^2-1)}\cosh\theta\}^{n+1}},$$

whence

$$\frac{\nu^2-1}{n}\frac{dQ_n}{d\nu} = \nu Q_n - Q_{n-1},$$

which proves (12). Eliminating $dQ_n/d\nu$ between (11) and (12) we obtain (10).

248. By employing either of the definite integral expressions (4) for a zonal harmonic, it can be shown that P_n satisfies (10), (11) and (12).

We obviously have

$$P_0 = 1, \quad P_1 = \nu, \quad P_2 = \tfrac{1}{2}(3\nu^2-1),$$

$$Q_0 = \tfrac{1}{2}\log\frac{\nu+1}{\nu-1}, \quad Q_1 = \nu Q_0 - 1.$$

249. We can now prove three more equations, viz.

$$P_{n+1}Q_n - P_nQ_{n+1} = \frac{1}{n+1} \quad\text{..................} (14),$$

$$P'_nQ_n - P_nQ'_n = \frac{1}{\nu^2-1} \quad\text{..................} (15),$$

$$P'_nQ'_{n+1} - P'_{n+1}Q'_n = \frac{n+1}{\nu^2-1} \quad\text{..................} (16),$$

where the accents denote differentiation with respect to ν.

From (10) it follows that

$$P_{n+1}Q_n - P_nQ_{n+1} = \frac{n}{n+1}(P_nQ_{n-1} - P_{n-1}Q_n),$$

$$= \frac{1}{n+1}(\nu Q_0 - Q_1),$$

$$= \frac{1}{n+1},$$

which proves (14); the other two equations can be established in a similar manner.

250. We shall now obtain an expression for Q_n^m in the form of a definite integral. Let

$$V_m = \int_0^\infty \frac{\sinh^{2m}\theta\, d\theta}{\{\nu + \sqrt{(\nu^2-1)}\cosh\theta\}^{n+m+1}}.$$

Then

$$\frac{dV_m}{d\nu} = -\frac{n+m+1}{\sqrt{(\nu^2-1)}} \int_0^\infty \frac{\{\sqrt{(\nu^2-1)} + \nu\cosh\theta\}\sinh^{2m}\theta\, d\theta}{\{\nu + \sqrt{(\nu^2-1)}\cosh\theta\}^{n+m+2}},$$

$$= -\frac{n+m+1}{\sqrt{(\nu^2-1)}} \int_0^\infty \frac{\sinh^{2m}\theta\cosh\theta\, d\theta}{\{\nu + \sqrt{(\nu^2-1)}\cosh\theta\}^{n+m+1}} + (n+m+1)\, V_{m+1}.$$

Integrating by parts, we find

$$\frac{dV_m}{d\nu} = -\frac{(n+m+1)(n-m)}{(2m+1)} V_{m+1}.$$

Now $V_0 = Q_n$, hence

$$\frac{d^mQ_n}{d\nu^m} = \frac{(-)^m (n-m+1)(n-m+2)\ldots(n+m)}{1\,.\,3\ldots.(2m-1)} V_m,$$

therefore

$$Q_n^m = \frac{(-)^m (n+m)!\,(\nu^2-1)^{\frac{1}{2}m}}{(n-m)!\,1\,.\,3\ldots(2m-1)} \int_0^\infty \frac{\sinh^{2m}\theta\, d\theta}{\{\nu + \sqrt{(\nu^2-1)}\cosh\theta\}^{n+m+1}} \ldots(17).$$

This expression is true for all positive values of m and n such that $n \overline{\overline{>}} m$.

251. We shall hereafter show that the potentials of ovary ellipsoids can always be expressed in terms of a series of P and Q functions; but in order to express the potentials of planetary ellipsoids in a similar manner, we require the functions which constitute the two solutions of the equation

$$\frac{d}{d\nu}(1+\nu^2)\frac{d\psi}{d\nu} + \frac{m^2\psi}{1+\nu^2} - n(n+1)\psi = 0 \ldots\ldots\ldots(18).$$

These two solutions may evidently be deduced from our previous results by putting $\iota\nu$ for ν, and rejecting imaginary factors. Beginning with the case of $m=0$, the complete solution of (18) is

$$AP_n(\iota\nu) + BQ_n(\iota\nu),$$

where

$$P_n(\iota\nu) = \frac{(-)^{\frac{1}{2}n}}{\pi} \int_0^\pi \{\nu + \sqrt{(\nu^2+1)}\cos\theta\}^n\, d\theta \ldots\ldots\ldots(19),$$

$$Q_n(\iota\nu) = (-)^{-\frac{1}{2}(n+1)} \int_0^\infty \frac{d\theta}{\{\nu + \sqrt{(\nu^2+1)}\cosh\theta\}^{n+1}} \ldots\ldots(20).$$

If therefore we denote the two definite integrals by $\pi p_n(\nu)$ and $q_n(\nu)$ respectively, the solution of the equation

$$\frac{d}{d\nu}(1+\nu^2)\frac{d\psi}{d\nu} - n(n+1)\psi = 0 \;\dots\dots\dots\dots (21),$$

may be written

$$\psi = Ap_n(\nu) + Bq_n(\nu).$$

252. From (19) and (20) we easily obtain

$$q_0 = \cot^{-1}\nu, \qquad q_1 = 1 - \nu\cot^{-1}\nu.$$
$$p_0 = 1, \qquad p_1 = \nu,$$

and we can show as in § 247 that,

$$\left.\begin{aligned}
&(n+2)\,q_{n+2} + (2n+3)\,\nu q_{n+1} - (n+1)\,q_n = 0\\
&\frac{\nu^2+1}{n+1}\frac{dq_n}{d\nu} = -q_{n+1} - \nu q_n,\\
&\frac{\nu^2+1}{n}\frac{dq_n}{d\nu} = \nu q_n - q_{n-1}.
\end{aligned}\right\}\dots\dots\dots (22).$$

The last three equations are also satisfied by $(-)^n p_n$; also

$$\left.\begin{aligned}
p_{n+1}q_n + q_{n+1}p_n &= \frac{1}{n+1}\\
p'_n q_n - q'_n p_n &= \frac{1}{\nu^2+1}\\
p'_n q'_{n+1} + q'_n p'_{n+1} &= \frac{n+1}{\nu^2+1}
\end{aligned}\right\}\dots\dots\dots\dots\dots\dots (23).$$

If we put $\cosh\theta = \sec\phi$, we obtain

$$q_n = \int_0^{\frac{1}{2}\pi} \frac{\cos^n\phi\, d\phi}{\{\nu\cos\phi + \sqrt{(\nu^2+1)}\}^{n+1}},$$

therefore

$$\left.\begin{aligned}
q_{2n}(0) &= \tfrac{1}{2}\pi H_n\\
q_{2n+1}(0) &= \frac{1}{(2n+1)H_n}
\end{aligned}\right\}\dots\dots\dots\dots\dots\dots (24),$$

where

$$H_n = \frac{1.3\dots 2n-1}{2^n\, n!}.$$

253. It can also be shown that if ψ be any solution of (21), then $(1+\nu^2)^{\frac{1}{2}m}\, d^m\psi/d\nu^m$ is a solution of (18); whence the complete integral of (18) may be written

$$Ap_n^m + Bq_n^m,$$

where

$$\left.\begin{aligned}
p_n^m &= (1+\nu^2)^{\frac{1}{2}m}\frac{d^m p_n}{d\nu^m}\\
q_n^m &= (1+\nu^2)^{\frac{1}{2}m}\frac{d^m q_n}{d\nu^m}
\end{aligned}\right\}\dots\dots\dots\dots\dots\dots (25).$$

Neumann's Transformation.

254. Having obtained these preliminary results, we shall now show by means of a transformation due to C. Neumann[1], that a solution of Laplace's equation can, in certain cases, be obtained in the form of a series $f(\xi)\,F(\eta)\,(m\phi+\alpha)$, where ξ and η are conjugate functions z and ϖ.

Laplace's equation when transformed into cylindrical coordinates z, ϖ and ϕ becomes

$$\frac{d^2V}{dz^2}+\frac{d^2V}{d\varpi^2}+\frac{1}{\varpi}\frac{dV}{d\varpi}+\frac{1}{\varpi^2}\frac{d^2V}{d\phi^2}=0\ldots\ldots\ldots\ldots\ldots(26).$$

Let $$V=V'\sin(m\phi+\alpha),$$

where V' is a function of z and ϖ only; substituting in (26), the equation for determining V' is

$$\frac{d^2V'}{dz^2}+\frac{d^2V'}{d\varpi^2}+\frac{1}{\varpi}\frac{dV'}{d\varpi}-\frac{m^2V'}{\varpi^2}=0.\ \ldots\ldots\ldots\ldots(27).$$

Let $$V'=U\varpi^{-\frac{1}{2}},$$

then (27) becomes

$$\frac{d^2U}{dz^2}+\frac{d^2U}{d\varpi^2}+\frac{1}{\varpi^2}(\tfrac{1}{4}-m^2)\,U=0\ \ldots\ldots\ldots\ldots(28).$$

Let $$z+\iota\varpi=f(\xi+\iota\eta),$$

$$J^2=\left(\frac{d\xi}{dz}\right)^2+\left(\frac{d\xi}{d\varpi}\right)^2=\left(\frac{d\eta}{dz}\right)^2+\left(\frac{d\eta}{d\varpi}\right)^2,$$

then (28) becomes

$$\frac{d^2U}{d\xi^2}+\frac{d^2U}{d\eta^2}+\frac{1}{J^2\varpi^2}(\tfrac{1}{4}-m^2)\,U=0.\ \ldots\ldots\ldots\ldots(29).$$

Now if $U=W\sqrt{uv}$ where u is a function of ξ alone and v is a function of η, (29) may be put into the form

$$\frac{1}{u}\frac{d}{d\xi}\left(u\frac{dW}{d\xi}\right)+\frac{1}{v}\frac{d}{d\eta}\left(v\frac{dW}{d\eta}\right)+\frac{1}{J^2\varpi^2}(\tfrac{1}{4}-m^2)\,W$$
$$+\left(\frac{u''}{2u}-\frac{u'^2}{4u^2}+\frac{v''}{2v}-\frac{v'^2}{4v^2}\right)W=0\ \ldots\ldots\ldots\ldots\ldots(30),$$

the accents denoting differentiation. From the form of the above

[1] *Theorie der Elektricitäts- und Wärme-Vertheilung in einem Ringe.* Halle, 1864.

equation, it follows that if $(J\varpi)^{-2}$ is either a function of ξ or η only, or the sum of two such functions, we can express W in a series of terms of the type $X_n Y_n$, where X_n is a function of ξ and Y_n is a function of η alone.

255. If we put

$$z + \iota\varpi = c \cos(\xi - \iota\eta),$$

then

$$z = c \cos \xi \cosh \eta,$$

$$\varpi = c \sin \xi \sinh \eta,$$

the equations $\eta = \beta$, $\xi = \alpha$ represent a family of confocal ovary ellipsoids and hyperboloids of two sheets respectively; also

$$J^{-2}\varpi^{-2} = \operatorname{cosec}^2 \xi + \operatorname{cosech}^2 \eta,$$

whence Neumann's transformation is applicable. Let

$$u = \sin \xi, \quad v = \sinh \eta,$$

$$\mu = \cos \xi, \quad \nu = \cosh \eta.$$

Then
$$W = U(c/\varpi)^{\frac{1}{2}} = V'c^{\frac{1}{2}},$$

and (30) becomes

$$\frac{d}{d\mu}(1-\mu^2)\frac{dV'}{d\mu} - \frac{d}{d\nu}(1-\nu^2)\frac{dV'}{d\nu} - \left(\frac{1}{1-\mu^2} - \frac{1}{1-\nu^2}\right)m^2 V' = 0.$$

This equation is satisfied by the series $\Sigma X_n Y_n$, where X_n and Y_n respectively satisfy the equations

$$\frac{d}{d\mu}(1-\mu^2)\frac{dX_n}{d\mu} + \left(C - \frac{m^2}{1-\mu^2}\right)X_n = 0,$$

$$\frac{d}{d\nu}(1-\nu^2)\frac{dY_n}{d\nu} + \left(C - \frac{m^2}{1-\nu^2}\right)Y_n = 0,$$

and C is some constant; hence

$$V = \Sigma\Sigma X_n Y_n \sin(m\phi + \alpha_m).$$

In order to determine the constant C, we observe that the potential at an external point of the ellipsoid $(z/a)^2 + (\varpi/c)^2 = 1$ is

$$V = \pi\rho a c^2 \int_\lambda^\infty \left(1 - \frac{\varpi^2}{c^2+\lambda} - \frac{z^2}{a^2+\lambda}\right)\frac{d\lambda}{(a^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)},$$

where λ is the positive root of the equation

$$\frac{z^2}{a^2+\lambda} + \frac{\varpi^2}{c^2+\lambda} = 1.$$

By § 148, equations (12) and (13), and by § 248 it is easily seen that each of the three integrals of which V is composed, are

respectively proportional to $Q_0(\nu)$, $Q_1^{\,1}(\nu)\,P_1^{\,1}(\mu)$ and $Q_1(\nu)\,P_1(\mu)$; whence $C = n(n+1)$ and the general value of V' is,

$$V' = \Sigma\,\{A_n P_n^m(\nu) + B_n\,Q_n^m(\nu)\}.$$

Along the line joining the foci, $\nu = 1$, and the Q functions are therefore infinite; on the other hand ν, and therefore the P functions, are infinite at infinity. But the Q functions and their derivatives do not become infinite at infinity, and J vanishes at infinity; also the P functions and their derivatives are finite and continuous along the line joining the foci; hence for space outside the ellipsoid

$$V = \Sigma\Sigma A_n Q_n^m(\nu)\,P_n^m(\mu)\sin(m\phi + \alpha_m),$$

and inside

$$V = \Sigma\Sigma B_n P_n^m(\nu)\,P_n^m(\mu)\sin(m\phi + \alpha_m),$$

but for space bounded by two confocal ellipsoids both functions may occur.

256. If we put

$$\varpi + \iota z = c\cos(\xi - \iota\eta),$$

the surfaces $\eta = \beta$, $\xi = \alpha$ will represent a family of confocal planetary ellipsoids and hyperboloids of one sheet; and if we put $\mu = \sin\xi$, $\nu = \sinh\eta$, it can be shown in a similar manner, that the potential at all points outside a planetary ellipsoid can be expressed in the form of the series

$$V = \Sigma\Sigma A_n q_n^m(\nu)\,P_n^m(\mu)\sin(m\phi + \alpha_m),$$

and at an internal point

$$V = \Sigma\Sigma B_n p_n^m(\nu)\,P_n^m(\mu)\sin(m\phi + \alpha_m).$$

257. We shall now give some examples.

Let a fixed ovary ellipsoid be immersed in an infinite liquid, and let the axes vary with the time, but so that the volume of the solid remains constant. If ϕ be the velocity potential, a and b the polar and equatorial semi-axes, and $c = (a^2 - b^2)^{\frac{1}{2}}$, the surface condition is

$$\frac{d\phi}{dn} = 2p\left(\frac{\dot{b}\varpi^2}{b^3} + \frac{\dot{a}z^2}{a^3}\right).$$

But $\qquad dn = acp^{-1}d\nu,$

and $\qquad \dot{a}/a + 2\dot{b}/b = 0.$

Therefore at the surface

$$\frac{d\phi}{d\nu} = \dot{a}c\,(3\mu^2 - 1) = 2\dot{a}cP_2\,(\mu).$$

Therefore:

$$\phi = 2\,\frac{\dot{a}cQ_2\,(\nu)\,P_2\,(\mu)}{Q'_2\,(\gamma)},$$

where $\gamma = a/c$ is the value of ν at the surface, and the accents denote differentiation.

In the corresponding case of a planetary ellipsoid,

$$\phi = 2\,\frac{\dot{a}cq_2\,(\nu)\,P_2\,(\mu)}{q_2'\,(\gamma)}.$$

258. When a solid of revolution is moving parallel to its axis with velocity V, we have shown in § 160 that if $\psi = \chi\varpi$, where ψ is Stokes' current function, χ is a solution of the equation

$$\frac{d^2\chi}{dz^2} + \frac{d^2\chi}{d\varpi^2} + \frac{1}{\varpi}\frac{d\chi}{d\varpi} - \frac{\chi}{\varpi^2} = 0,$$

whence in the case of an ovary ellipsoid

$$\psi = \varpi\Sigma_1^\infty A_n Q_n^{\,1}\,(\nu)\,P_n^{\,1}\,(\mu),$$

and in the case of a planetary ellipsoid

$$\psi = \varpi\Sigma_1^\infty A_n q_n^{\,1}\,(\nu)\,P_n^{\,1}\,(\mu).$$

Now for motion parallel to the axis, the surface condition is

$$\psi = \tfrac{1}{2}V\varpi^2,$$

also at the surface

$$\begin{aligned}\varpi &= c\,(\gamma^2 - 1)^{\frac{1}{2}}\,(1 - \mu^2)^{\frac{1}{2}},\\ &= bP_1^{\,1}\,(\mu),\end{aligned}$$

whence
$$\psi = \tfrac{1}{2}Vb\varpi\,\frac{Q_1^{\,1}\,(\nu)\,P_1^{\,1}\,(\mu)}{Q_1^{\,1}\,(\gamma)};$$

and in the case of a planetary ellipsoid

$$\psi = \tfrac{1}{2}Vb\varpi\,\frac{q_1^{\,1}\,(\nu)\,P_1^{\,1}\,(\mu)}{q_1^{\,1}\,(\gamma)}.$$

259. If
$$z + \iota\varpi = c\sec(\xi + \iota\eta),$$

the surface $\eta =$ const., is the inverse of an ovary ellipsoid with respect to its centre, also

$$J^{-2}\varpi^{-2} = \operatorname{cosec}^2\xi + \operatorname{cosech}^2\eta.$$

In Neumann's transformation put

$$u = \sin \xi, \quad v = \sinh \eta,$$
$$\mu = \cos \xi, \quad \nu = \cosh \eta,$$

and the equation to be satisfied by W, becomes

$$\frac{d}{d\mu}(1-\mu^2)\frac{dW}{d\mu} - \frac{d}{d\nu}(1-\nu^2)\frac{dW}{d\nu} - \left(\frac{1}{1-\mu^2} - \frac{1}{1-\nu^2}\right) m^2 W = 0.$$

Also $$W = U(uv)^{-\frac{1}{2}} = V'rc^{-\frac{1}{2}}.$$

Whence remembering that the lines $\nu = 1$ lie outside the surface, and that $\nu = \infty$ at the centre; the value of V at an external point will be

$$V = \frac{c}{r} \Sigma\Sigma A_n \frac{P_n^m(\nu) P_n^m(\mu)}{P_n^m(\gamma)} \sin(m\phi + \alpha_m),$$

and at an internal point

$$V = \frac{c}{r} \Sigma\Sigma A_n \frac{Q_n^m(\nu) P_n^m(\mu)}{Q_n^m(\gamma)} \sin(m\phi + \alpha_m).$$

260. The value of the current function ψ, at an external point will be

$$\psi = \frac{c\varpi}{r} \Sigma_1^\infty A_n \frac{P_n^1(\nu) P_n^1(\mu)}{P_n^1(\gamma)}.$$

If therefore the solid be moving parallel to its axis with velocity V, the boundary condition becomes

$$Vr\varpi/2c = \Sigma_1^\infty A_n P_n^1(\mu),$$

we have therefore to find the expansion of $r\varpi$.

From the equations,

$$cz/r^2 = \mu\nu, \quad c\varpi/r^2 = (1-\mu^2)^{\frac{1}{2}}(\nu^2-1)^{\frac{1}{2}},$$

we obtain,

$$\frac{rz}{c^2} = \frac{\mu\nu}{(\mu^2+\nu^2-1)^{\frac{3}{2}}}, \quad \frac{r\varpi}{c^2} = \sqrt{\frac{(\nu^2-1)(1-\mu^2)}{(\mu^2+\nu^2-1)^3}}.$$

Since z is a potential function, rz can be expanded in a series of spheroidal harmonics, and since only odd powers of ν can occur, we must have

$$\mu\nu(\mu^2+\nu^2-1)^{-\frac{3}{2}} = \Sigma_0^\infty B_{2n+1} Q_{2n+1}(\nu) P_{2n+1}(\mu).$$

Therefore

$$\frac{2B_{2n+1}Q_{2n+1}}{4n+3} = \frac{1}{\nu^2}\int_{-1}^{1} \mu \left\{1 + \ldots (2s+1) H_s \frac{(1-\mu^2)^s}{\nu^{2s}}\right\} P_{2n+1}\, d\mu,$$

where
$$H_s = \frac{1.3\ldots(2s-1)}{2^s . s!}.$$

Let
$$U_s = \int_{-1}^{1} \mu (1-\mu^2)^s P_{2n+1}\, d\mu.$$

Then
$$(2n+1)(2n+2)\, U_s = -\int_{-1}^{1} \mu (1-\mu^2)^s \frac{d}{d\mu}(1-\mu^2) \frac{dP_{2n+1}}{d\mu}\, d\mu,$$
$$= (2s+1)\int_{-1}^{1} (1-\mu^2)^{s+1} \frac{dP_{2n+1}}{d\mu}\, d\mu$$
$$- 2s\int_{-1}^{1} (1-\mu^2)^{s} \frac{dP_{2n+1}}{d\mu}\, d\mu,$$
$$= (2s+1)(2s+2)\, U_s - 2^2 s^2 U_{s-1}.$$

Therefore
$$U_s = \frac{2s^2}{(s-n)(2s+2n+3)}\, U_{s-1},$$
$$= \frac{2^r s^2 (s-1)^2 \ldots (s-r+1)^2\, U_{s-r}}{(s-n)(s-n-1)\ldots(s-n-r+1)(2s+2n+3)(2s+2n+1)\ldots(2s+2n-2r+5)}$$

Now
$$U_n = (-)^n \int_{-1}^{1} \mu^{2n+1} P_{2n+1}\, d\mu$$
$$= \frac{(-)^n\, 2^{n+1}\, n!}{(2n+3)(2n+5)\ldots(4n+3)}.$$

Therefore if
$$s = n + r,$$
$$(2n+2r+1)\, H_{n+r} U_{n+r} = \frac{2(-)^n 1.3\ldots(2n+2r+1)(n+1)(n+2)\ldots(n+r)}{r!\,(2n+3)(2n+5)\ldots(4n+2r+3)}.$$
$$= \frac{2(-)^n(2n+1) H_n (2r+1)(2r+2)\ldots(2n+2r+1)}{(2r+1)(2r+3)\ldots(4n+2r+3)}.$$

Therefore $\quad 2B_{2n+1}\, Q_{2n+1}/(4n+3) = 2(-)^n (2n+1)\, H_n Q_{2n+1}$,

whence
$$rz/c^2 = \mu\nu\, (\mu^2+\nu^2-1)^{-\frac{3}{2}} = \Sigma_0^\infty (-)^n (2n+1)(4n+3)\, H_n Q_{2n+1} P_{2n+1}.$$

Integrating both sides with respect to ν, we obtain
$$\mu\, (\mu^2+\nu^2-1)^{-\frac{1}{2}} = \Sigma\, B_{2n+1} P_{2n+1} \int_\nu^\infty Q_{2n+1}\, d\nu,$$
$$= -\Sigma\, B_{2n+1} P_{2n+1} \frac{\sqrt{(\nu^2-1)}\, Q^1_{2n+1}}{(2n+1)(2n+2)}.$$

Differentiating with respect to μ, and multiplying by $\sqrt{(1-\mu^2)}$, we obtain
$$\frac{(1-\mu^2)^{\frac{1}{2}} (\nu^2-1)^{\frac{1}{2}}}{(\mu^2+\nu^2-1)^{\frac{3}{2}}} = -\Sigma\, (-)^n \frac{4n+3}{2n+2} H_n\, Q^1_{2n+1} P^1_{2n+1}.$$

Whence

$$\psi = -\frac{Vc^2\varpi}{2r}\Sigma_0^\infty \frac{(-)^n (4n+3)\, Q^1_{2n+1}(\gamma)}{(2n+2)\, P^1_{2n+1}(\gamma)} H_n P^1_{2n+1}(\nu)\, P^1_{2n+1}(\mu).$$

261. By proceeding in a similar manner, it will be found that when the solid is the inverse of a planetary ellipsoid with respect to its centre, the value of ψ is[1]

$$\psi = -\frac{Vc^2\varpi}{2r}\Sigma_0^\infty \frac{(4n+3)\, q^1_{2n+1}(\gamma)}{(2n+2)\, p^1_{2n+1}(\gamma)} H_n\, p^1_{2n+1}(\nu)\, P^1_{2n+1}(\mu).$$

262. By making use either of the method of inversion or the transformation,

$$z + \iota\varpi = 2c \sec^2 \tfrac{1}{2}(\xi + \iota\eta),$$

the same problem can be solved when the meridian curve is an elliptic limaçon, i.e. the inverse of an ellipse with respect to its focus[2].

Bessel's Functions.

263. The properties of the Bessel's function $J_m(x)$ where m is any positive integer, are so fully discussed in *Todhunter's Functions of Laplace, Lamé and Bessel,* and *Lord Rayleigh's Treatise on Sound,* that it will be unnecessary to consider them in the present chapter, farther than to note that $J_m(x)$ satisfies the differential equation

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \left(1 - \frac{m^2}{x^2}\right) u = 0 \quad \dots\dots\dots\dots\dots (31),$$

and that it can be expressed either in the form of the definite integral

$$J_m(x) = \frac{x^m}{\pi\, 1.3\dots(2m-1)} \int_0^\pi \cos(x\cos\phi) \sin^{2m}\phi\, d\phi,$$

or by means of the series

$$J_m(x) = \frac{x^m}{2^m m!}\left\{1 - \frac{x^2}{2(2m+2)} + \frac{x^4}{2.4(2m+2)(2m+4)} - \frac{x^6}{2.4.6(2m+2)(2m+4)(2m+6)} + \dots\right\}.$$

[1] *Quarterly Journal,* vol. XIX. pp. 368—370.

[2] *Proc. Camb. Phil. Soc.* vol. VI. p. 8.

We shall also prove the following theorem which is analogous to Fourier's theorem, by means of which a given function can be expressed in the form of a definite integral involving Bessel's functions.

264. *If p and q be any positive real quantities, and $\phi(\varpi)$ is a function which is finite and continuous for all values of ϖ which lie between the limits p and q, but which is not necessarily finite at the limits, then the definite integral*

$$\int_0^\infty d\lambda \int_q^p \lambda u \phi(u) J_m(\lambda u) J_m(\lambda \varpi) du \quad \text{............(32)},$$

is equal to $\phi(\varpi)$ when ϖ lies between the limits p and q, and is equal to zero when ϖ lies beyond these limits.

In order to prove the theorem, consider a thin plane conductor bounded by two concentric circles of radii p and q, which is electrified in such a manner that the density on either side is equal to

$$\tfrac{1}{2}\phi(\varpi)\cos m\phi.$$

The potential will be

$$V = \int_q^p \int_\phi^{2\pi+\phi} \frac{u\phi(u)\cos m\phi' du d\phi'}{\{z^2 + \varpi^2 + u^2 - 2\varpi u \cos(\phi' - \phi)\}^{\frac{1}{2}}}.$$

Let
$$\phi' - \phi = \eta$$
$$R^2 = \varpi^2 + u^2 - 2\varpi u \cos\eta.$$

Then $$V = \int_q^p \int_0^{2\pi} \frac{u\phi(u)(\cos m\phi \cos m\eta - \sin m\phi \sin m\eta)\, du d\eta}{(z^2 + R^2)^{\frac{1}{2}}}.$$

The second integral vanishes; also since

$$\int_0^\infty \epsilon^{-\lambda z} J_0(\lambda R)\, d\lambda = (z^2 + R^2)^{-\frac{1}{2}}$$

the first is equal to

$$2\cos m\phi \int_0^\infty d\lambda \int_q^p du \int_0^\pi \epsilon^{-\lambda z} u\phi(u) \cos m\eta J_0(\lambda R)\, d\eta.$$

Now[1] $J_0(\lambda R) = J_0(\lambda\varpi) J_0(\lambda u) + 2\Sigma_1^\infty J_m(\lambda\varpi) J_m(\lambda u) \cos m\eta$,

whence $$V = 2\pi \cos m\phi \int_0^\infty d\lambda \int_q^p \epsilon^{-\lambda z} u\phi(u) J_m(\lambda u) J_m(\lambda\varpi)\, du.$$

The density $$= -\frac{1}{4\pi}\left(\frac{dV}{dz}\right)_0,$$

[1] Todhunter, *Functions of Laplace &c.* § 453.

hence this quantity must be equal to $\frac{1}{2}\phi(\varpi)\cos m\phi$ when $p > \varpi > q$, and must be zero when ϖ lies beyond the limits p and q, whence

$$\int_0^\infty d\lambda \int_q^p \lambda u \phi(u) J_m(\lambda u) J_m(\lambda\varpi)\, du = \phi(\varpi),\ p > \varpi > q$$
$$= 0 \qquad \left.\begin{matrix}\varpi > p \\ \text{or } \varpi < q\end{matrix}\right\}.$$

265. If a charged conductor of the form which we are considering is placed in a field of force, the density will usually be infinite at the edges, but dV/dz will always be finite except at the edges; whence although it is necessary that $\phi(\varpi)$ should be finite and continuous between the limits p and q, it is not in general necessary that it should be finite at the limits. There are however two special cases, viz. (i) $q = 0$, p finite; and (ii) $p = \infty$, q finite, which require separate consideration.

The first case is that of a circular disc of radius p; and if $\phi(\varpi)$ became infinite when $\varpi = 0$, there would be a singular point at the origin.

The second case is that of an infinite plane screen having a circular aperture, and if $\phi(\varpi)$ became infinite when $\varpi = \infty$, the density would be infinite at an infinite distance from the aperture, which seems to be physically impossible.

If therefore in the first case $\phi(\varpi) = \infty$ when $q = 0$; and in the second case $\phi(\varpi) = \infty$ when $p = \infty$, the theorem could not be safely employed.

If $\phi(\varpi)$ is finite and continuous for all values of ϖ between 0 and ∞ inclusive, we may put $p = \infty$, $q = 0$, and the theorem becomes

$$\phi(\varpi) = \int_0^\infty d\lambda \int_0^\infty \lambda u \phi(u) J_m(\lambda u) J_m(\lambda\varpi)\, du \quad\ldots\ldots(33)$$

for all positive values of ϖ.

266. We must now consider a class of functions analogous to Bessel's functions, which are obtained by changing x into ιx.

Putting $x = \iota x$, (31) becomes

$$\frac{d^2 u}{dx^2} + \frac{1}{x}\frac{du}{dx} - \left(1 + \frac{m^2}{x^2}\right) u = 0 \quad\ldots\ldots\ldots\ldots\ldots (34).$$

This equation, as we shall proceed to show, has two independent integrals, one of which is finite or zero when $x = 0$, and is infinite

when $x=\infty$; and the other is infinite when $x=0$ and zero when $x=\infty$. We shall denote these two solutions by the symbols $I_m(x)$ and $K_m(x)$ respectively.

The function I_m is derived from J_m by changing x into ιx and rejecting imaginary factors; we thus obtain

$$I_m(x)=\frac{x^m}{\pi\, 1.3\ldots(2m-1)}\int_0^\pi \cosh(x\cos\phi)\sin^{2m}\phi\, d\phi \ldots(35),$$

or as a series

$$I_m(x)=\frac{x^m}{2^m m!}\left\{1+\frac{x^2}{2(2m+2)}+\frac{x^4}{2.4(2m+2)(2m+4)}+\ldots\right\} (36).$$

267. In (34) put $u=x^m v_m$, and we obtain

$$\frac{d^2v_m}{dx^2}+\frac{2m+1}{x}\frac{dv_m}{dx}-v_m=0,$$

in this put $x^2=y$, and we obtain

$$y\frac{d^2v_m}{dy^2}+(m+1)\frac{dv_m}{dy}-\tfrac{1}{4}v_m=0.$$

Differentiating with respect to y, we obtain

$$\frac{dv_m}{dy}=v_{m+1}.$$

Hence if u_0 denote any solution of the equation

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}-u=0 \ldots\ldots\ldots\ldots(37),$$

a solution of (34) will be

$$u_m=x^m\left\{\frac{d}{d(x^2)}\right\}^m u_0 \ldots\ldots\ldots\ldots(38).$$

If therefore the value of K_0 is known, the value of K_m can be obtained by means of (38).

268. Perhaps the simplest way of determining K_0 is derived from the consideration that Bessel's functions are limiting forms of spheroidal harmonics. Let $c\nu$ be the major axis of an ovary ellipsoid, and let

$$c\sqrt{(\nu^2-1)}=r, \qquad n(n+1)=\lambda^2c^2,$$

then if c and n increase indefinitely, whilst ν approaches indefinitely near to unity, but so that both r and λ remain finite, the ellipsoid ultimately becomes a circular cylinder.

In equation (7) change the variable from ν to r and we obtain

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \frac{1}{c^2}\left(r^2 \frac{d^2u}{dr^2} + 2r\frac{du}{dr}\right) - \frac{n(n+1)\,u}{c^2} = 0,$$

which ultimately becomes

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \lambda^2 u = 0,$$

which agrees with (37) if $\lambda r = x$.

Also
$$Q_n = \int_0^\infty \frac{d\theta}{\{\nu + \sqrt{(\nu^2-1)}\cosh\theta\}^{n+1}}$$
$$= \int_0^\infty \exp\{-(n+1)\log[\nu + \sqrt{(\nu^2-1)}\cosh\theta]\}\,d\theta.$$

Now $(n+1)\log\{\nu + \sqrt{(\nu^2-1)}\cosh\theta\}$
$$= \tfrac{1}{2}\{1 + \sqrt{(1+4\lambda^2c^2)}\}\log\{(1+r^2/c^2)^{\frac{1}{2}} + r/c\,.\cosh\theta\}$$
$$= \lambda r\cosh\theta,$$

ultimately; hence the limiting form of $Q_n(\nu)$ is

$$Q_n = \int_0^\infty \epsilon^{-\lambda r\cosh\theta}\,d\theta;$$

whence it follows that

$$K_0(x) = \int_0^\infty \epsilon^{-x\cosh\theta}\,d\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots (39).$$

Since $K_0(x)$ is infinite when $x=0$, it is evidently the solution we require.

Another form of K_0 may be obtained by means of the integral

$$\int_0^\infty \frac{\cos\lambda v\,dv}{a^2+v^2} = \frac{\pi}{2a}\epsilon^{-\lambda a},$$

for putting $z = \sinh\theta$ in (39) we obtain

$$K_0 = \int_0^\infty \frac{\epsilon^{-x\sqrt{(1+z^2)}}\,dz}{\sqrt{(1+z^2)}}$$
$$= \frac{2}{\pi}\int_0^\infty\int_0 \frac{\cos x\phi\,d\phi\,dz}{1+\phi^2+z^2}$$
$$= \int_0^\infty \frac{\cos x\phi\,d\phi}{(1+\phi^2)^{\frac{1}{2}}}$$
$$= \int_0^\infty \frac{\cos\chi\,d\chi}{(x^2+\chi^2)^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots (40).$$

Whence by (38)

$$K_m = \frac{(-)^m 1.3\ldots(2m-1)}{2^m x^m} \int_0^\infty \frac{\cos x\phi d\phi}{(1+\phi^2)^{\frac{1}{2}(2m+1)}} \ldots\ldots(41).$$

Also $$K_0' = 2K_1 \ldots\ldots\ldots\ldots\ldots\ldots(42).$$

269. By means of the integral

$$\int_0^\infty \epsilon^{-\phi u} J_0(u)\, du = (1+\phi^2)^{-\frac{1}{2}},$$

we obtain

$$K_0 = \int_0^\infty \frac{\cos x\phi d\phi}{(1+\phi^2)^{\frac{1}{2}}} = \int_0^\infty \int_0^\infty \epsilon^{-\phi u} J_0(u) \cos x\phi d\phi du$$

$$= \int_0^\infty \frac{u J_0(u)\, du}{x^2+u^2} \ldots\ldots\ldots\ldots\ldots\ldots(43),$$

$$= \int_0^\infty \frac{\alpha J_0(\alpha x)\, d\alpha}{1+\alpha^2} \ldots\ldots\ldots\ldots\ldots\ldots(44).$$

270. We shall now apply the preceding results, to determine the current function due to the motion parallel to its axis of the surface formed by the revolution of a cardioid about its axis.

If $$\xi + \iota\eta = c^{\frac{1}{2}} / (z - \iota\varpi)^{\frac{1}{2}}$$
we obtain $$\xi = (c/r)^{\frac{1}{2}} \cos \tfrac{1}{2}\theta, \qquad \eta = (c/r)^{\frac{1}{2}} \sin \tfrac{1}{2}\theta$$
and the surfaces $\xi = \alpha$, $\eta = \beta$ are the surfaces formed by the revolution of a cardioid about its axis. Also

$$J^{-2}\varpi^{-2} = \xi^{-2} + \eta^{-2}$$

hence Neumann's transformation can be employed.

In § 254 put $u = \xi, v = \eta$, and (30) becomes

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dW}{d\xi}\right) + \frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{dW}{d\eta}\right) - m^2\left(\frac{1}{\xi^2}+\frac{1}{\eta^2}\right)W = 0,$$

and this equation is satisfied if

$$W = \{A_m I_m(\lambda\eta) + B_m K_m(\lambda\eta)\} J_m(\lambda\xi)$$

where λ is undetermined. Also

$$V' = Wr^{-1}(\tfrac{1}{2}c)^{\frac{1}{2}}$$

whence $V = cr^{-1}\Sigma\{A_m I_m(\lambda\eta) + B_m K_m(\lambda\eta)\} J_m(\lambda\xi) \sin(m\phi + \alpha_m)$.

The preceding value of V is a suitable expression for determining the potential of the surface $\eta = \text{const.}$

Now a cardioid is the inverse of a parabola, and a parabola is a limiting form of an ellipse; and since the Q functions are suitable for space outside an ovary ellipsoid, and the P functions for space inside, it follows that the K functions are suitable for space outside a paraboloid of revolution, and the I functions for space inside. Hence the I functions are suitable for space *outside* the surface formed by the revolution of a cardioid, and the K functions for space *inside*; moreover the conditions of the problem do not enable us to assign any value to λ, and we must therefore give it all values from ∞ to 0, and replace the summation with respect to λ by a definite integral. Hence the potential outside the surface formed by the revolution of a cardioid is of the form

$$V = cr^{-1}\Sigma \sin(m\phi + \alpha_m)\int_0^\infty F(\lambda)\, I_m(\lambda\eta)\, J_m(\lambda\xi)\, d\lambda,$$

and inside

$$V = cr^{-1}\Sigma \sin(m\phi + \alpha_m)\int_0^\infty F(\lambda)\, K_m(\lambda\eta)\, J_m(\lambda\xi)\, d\lambda.$$

271. When the surface formed by the revolution of the cardioid $(r/c)^{\frac{1}{2}} = \sin\frac{1}{2}\theta$ or $\eta = 1$, is moving parallel to its axis with velocity V, the value of ψ may be written

$$\psi = \frac{c\varpi}{r}\int_0^\infty F(\lambda)\frac{I_1(\lambda\eta)\, J_1(\lambda\xi)}{I_1(\lambda)}\, d\lambda,$$

where $F(\lambda)$ has to be determined from the surface condition

$$Vr\varpi/2c = \int_0^\infty F(\lambda)\, J_1(\lambda\xi)\, d\lambda.$$

Now when $\eta = 1$,

$$r\varpi/c^2 = 2\xi/(1+\xi)^3$$

$$= 2\int_0^\infty\int_0^\infty \lambda\alpha^2(1+\alpha^2)^{-3} J_1(\lambda\alpha)\, J_1(\lambda\xi)\, d\alpha\, d\lambda.$$

By (43), $$K_0(\lambda) = \int_0^\infty \theta J_0(\theta)(\lambda^2+\theta^2)^{-1}\, d\theta.$$

Therefore $$2K_1 = K'_0 = -2\lambda\int_0^\infty \theta J_0(\theta)(\lambda^2+\theta^2)^{-2}\, d\theta.$$

Also $$\theta J_0 = J_1 + \theta J'_1.$$

Therefore $$K_1 = -\lambda\int_0^\infty (J_1 + \theta J'_1)(\lambda^2+\theta^2)^{-2}\, d\theta.$$

Integrating the last term by parts, and then putting $\theta = \lambda\alpha$, we obtain

$$K_1(\lambda) = -4\lambda^{-2}\int_0^\infty \alpha^2 J_1(\lambda\alpha)(1+\alpha^2)^{-3}\,d\alpha.$$

Therefore $\dfrac{r\varpi}{c^2} = \tfrac{1}{2}\displaystyle\int_0^\infty \lambda^3 K_1(\lambda)\,J_1(\lambda\xi)\,d\lambda,$

whence $F(\lambda) = -\tfrac{1}{4}Vc\lambda^3 K_1(\lambda),$

and
$$\psi = -\frac{Vc^2\varpi}{4r}\int_0^\infty \frac{\lambda^3 K_1(\lambda)}{I_1(\lambda)}\,I_1(\lambda\eta)\,J_1(\lambda\xi)\,d\lambda.$$

This expression, as well as the corresponding expression for the current function due to the motion of the inverse of an ellipsoid of revolution with respect to its centre, is of such an exceedingly complicated character, that it does not seem probable that progress is to be looked for in the direction of new surfaces of the third and higher orders.

Toroidal Functions[1].

272. The system of conjugate functions

$$x + \iota y = a\tan\tfrac{1}{2}(\xi + \iota\eta) \quad\ldots\ldots\ldots\ldots\ldots\ldots(45),$$

has been discussed in § 120, and it is there shown that the curves $\eta = \text{const.}$, represent the system of circles

$$x^2 + y^2 - 2ay\coth\eta + a^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots(46).$$

The centres of each circle of this system lie on the axis of y, and none of the circles cut the axis of x. If therefore we put

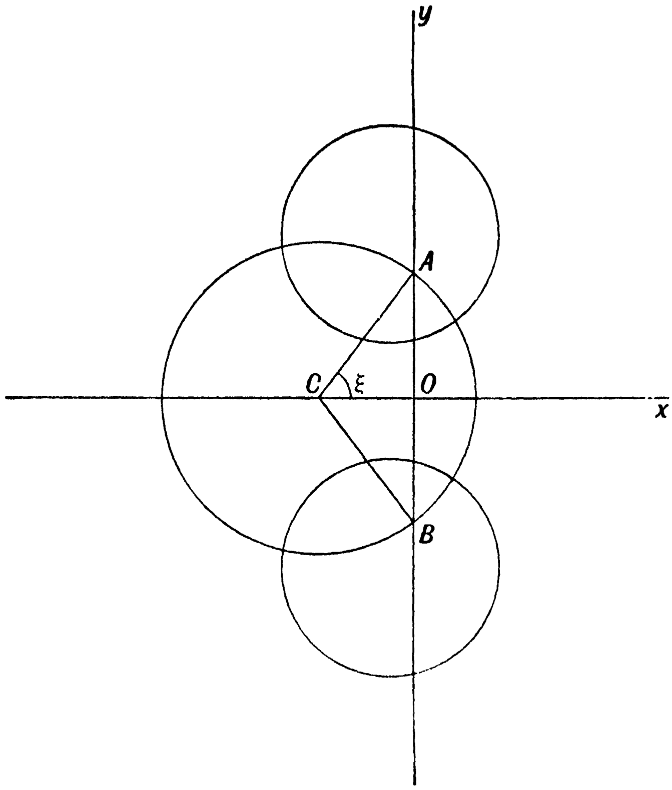
$$z + \iota\varpi = a\tan\tfrac{1}{2}(\xi + \iota\eta) \quad\ldots\ldots\ldots\ldots\ldots\ldots(47),$$

(46) becomes

$$z^2 + \varpi^2 - 2a\varpi\coth\eta + a^2 = 0,$$

which is the equation of a family of anchor rings or tores, whose common axis is the axis of z. When $\eta = \infty$, the tores degenerate into the circle formed by the revolution of the points A and B. This circle is called the *critical circle*.

[1] Hicks; *Phil. Trans.* 1881, p. 609: *Ibid.* 1884, p. 161.



Also since $J\varpi = \sinh \eta$, Neumann's transformation applies; if therefore we put $u = 1$, $v = \sinh \eta$, $\nu = \cosh \eta$ in (30) the equation for W becomes

$$\frac{d^2 W}{d\xi^2} + \frac{d}{d\nu} (\nu^2 - 1) \frac{dW}{d\nu} - \frac{m^2 W}{\nu^2 - 1} + \tfrac{1}{4} W = 0 \quad \text{.........(48)}.$$

Now W must evidently be periodic with respect to ξ, and must therefore be of the form $\Sigma \chi_n \cos (n\xi + \alpha_n)$ where n is a positive integer, and χ_n is a function of η alone. Substituting in (48) we obtain

$$\frac{d}{d\nu} (1 - \nu^2) \frac{d\chi_n}{d\nu} - \frac{m^2 \chi_n}{1 - \nu^2} + (n^2 - \tfrac{1}{4}) \chi_n = 0 \quad \text{........ (49)},$$

whence

$$V = (\cosh \eta + \cos \xi)^{\frac{1}{2}} \Sigma\Sigma \chi_n \cos (n\xi + \alpha_n) \sin (m\phi + \beta_m) \ldots (50).$$

The two integrals of (49) are called *Toroidal Functions,* and will be employed in Chapter XIV. in the discussion of circular vortices.

Equation (49) shows that χ_n is an associated function of degree $n - \frac{1}{2}$ and order m; but it will not be necessary to enter into the general discussion of this equation for all values of m, since in the hydrodynamical applications which follow, the functions of orders zero and unity are the only ones required. We shall begin with

the case of $m=0$, and show that if in the definite integral expressions for the two kinds of zonal harmonics, n be changed into $n-\frac{1}{2}$, the resulting integrals constitute two independent integrals of (49), one of which is finite when $\nu=1$ and infinite when $\nu=\infty$; and the other is infinite when $\nu=1$ and zero when $\nu=\infty$.

273. If in (49) we put $m=0$, we obtain

$$\frac{d}{d\nu}(1-\nu^2)\frac{d\chi_n}{d\nu}+(n^2-\tfrac{1}{4})\chi_n=0 \quad\text{............ (51)},$$

which is the equation satisfied by zonal toroidal functions. Writing for brevity C and S for $\cosh\eta$ and $\sinh\eta$, we know that the zonal harmonic of degree n of the first kind is expressible (omitting the factor π^{-1}) in either of the forms

$$\int_0^\pi (C+S\cos\theta)^n\,d\theta \text{ or } \int_0^\pi (C+S\cos\theta)^{-n-1}\,d\theta,$$

the second of which can be deduced from the first by means of the transformation $(C+S\cos\theta)(C+S\cos\theta')=1$. Similarly if we put

$$P_n=\int_0^\pi (C+S\cos\theta)^{\frac{1}{2}(2n-1)}\,d\theta \quad\text{............ (52)},$$

it can be shown by means of the same transformation that

$$P_n=\int_0^\pi (C+S\cos\theta)^{-\frac{1}{2}(2n+1)}\,d\theta \quad\text{........... (53)}.$$

We shall now show that either of the definite integrals (52) or (53) is a solution of (51).

From (52) we obtain

$$\frac{dP_n}{d\nu}=\tfrac{1}{2}(2n-1)\int_0^\pi (C+S\cos\theta)^{\frac{1}{2}(2n-3)}\left(1+\frac{C}{S}\cos\theta\right)d\theta.$$

Therefore

$$\begin{aligned} S^2\frac{dP_n}{d\nu} &= \tfrac{1}{2}(2n-1)\int_0^\pi (C+S\cos\theta)^{\frac{1}{2}(2n-3)}\{C(C+S\cos\theta)-1\}\,d\theta \\ &= \tfrac{1}{2}(2n-1)(CP_n-P_{n-1}) \quad\text{.............................. (54)}, \end{aligned}$$

and from (53)

$$\begin{aligned} S^2\frac{dP_n}{d\nu} &= -\tfrac{1}{2}(2n+1)\int_0^\pi \frac{C(C+S\cos\theta)-1}{(C+S\cos\theta)^{\frac{1}{2}(2n+3)}}\,d\theta \\ &= -\tfrac{1}{2}(2n+1)(CP_n-P_{n+1}) \quad\text{................. (55)}. \end{aligned}$$

Differentiating (54) with respect to ν, we obtain

$$\frac{d}{d\nu}\left(S^2\frac{dP_n}{d\nu}\right)=\tfrac{1}{2}(2n-1)\left(P_n+C\frac{dP_n}{d\nu}-\frac{dP_{n-1}}{d\nu}\right)$$

$$=\tfrac{1}{2}(2n-1)\{P_n+\tfrac{1}{2}(2n-1)CS^{-2}(CP_n-P_{n-1})$$

$$-\tfrac{1}{2}(2n-1)S^{-2}(P_n-CP_{n-1})\}$$

$$=(n^2-\tfrac{1}{4})P_n,$$

or

$$\frac{d}{d\nu}(1-\nu^2)\frac{dP_n}{d\nu}+(n^2-\tfrac{1}{4})P_n=0,$$

which shows that the definite integrals (52) and (53) are solutions of (51).

Eliminating $dP_n/d\nu$ from (54) and (55) we obtain the sequence equation

$$(2n+1)P_{n+1}-4nCP_n+(2n-1)P_{n-1}=0 \quad\ldots\ldots (56).$$

Equations (54), (55) and (56) are what equations (12), (11) and (10) become when n is changed into $n-\tfrac{1}{2}$.

From (52) it appears that $P_n=\infty$ when C i.e. $\nu=\infty$; and therefore $P_n=\infty$ when $\eta=\infty$; also when $\eta=0$, $C=1$, $S=0$ and $P_n=\pi$.

274. Again let

$$k^2=\frac{1}{(C+S)^2}=\epsilon^{-2\eta},$$

$$k'^2=\frac{2S}{C+S}=1-\epsilon^{-2\eta}.$$

Then

$$P_0=\int_0^\pi\frac{d\theta}{(C+S\cos\theta)^{\frac{1}{2}}}=2k^{\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{(1-k'^2\sin^2\phi)^{\frac{1}{2}}}=2k^{\frac{1}{2}}F' \ldots (57),$$

and

$$P_1=\int_0^\pi(C+S\cos\theta)^{\frac{1}{2}}d\theta=2k^{-\frac{1}{2}}\int_0^{\frac{1}{2}\pi}(1-k'^2\sin^2\phi)^{\frac{1}{2}}d\phi=2k^{-\frac{1}{2}}E' (58),$$

where F' and E' are the first and second complete elliptic integrals to mod. k'.

Having obtained the values of P_0 and P_1, the values of the successive functions can be calculated by means of the sequence equation (56).

275. We have shown that the zonal harmonic of the second kind is expressible in the form

$$\int_0^\infty \frac{d\theta}{(C+S\cosh\theta)^{n+1}},$$

and if we put

$$Q_n = \int_0^\infty \frac{d\theta}{(C+S\cosh\theta)^{\frac{1}{2}(2n+1)}},$$

it can be shown, as in the case of the P functions, that the above definite integral is a solution of (51). Also when $C=\infty$, $Q_n=0$; and when $C=1$ or $\eta=0$, $Q_n=\infty$. Hence the two functions P and Q constitute two independent integrals of (51). It can also be shown that the above value of Q_n satisfies equations (54), (55) and (56).

Again,

$$Q_0 = \int_0^\infty \frac{d\theta}{(C+S\cosh\theta)^{\frac{1}{2}}},$$

$$Q_1 = \int_0^\infty \frac{d\theta}{(C+S\cosh\theta)^{\frac{3}{2}}}.$$

In these change θ into $2\theta'$, and then put $\cosh\theta' = \sec\phi$; then $d\theta' = \sec\phi d\phi$, also when $\theta'=0$ or ∞, $\phi=0$ or $\frac{1}{2}\pi$; therefore

$$\begin{aligned} Q_0 &= 2\int_0^\infty \frac{d\theta'}{(C-S+2S\cosh^2\theta')^{\frac{1}{2}}}, \\ &= 2\int_0^{\frac{1}{2}\pi} \frac{d\phi}{\{C+S-(C-S)\sin^2\phi\}^{\frac{1}{2}}}, \\ &= 2k^{\frac{1}{2}}F \qquad\qquad (59). \end{aligned}$$

And

$$\begin{aligned} Q_1 &= 2\int_0^\infty \frac{d\theta'}{(C-S+2S\cosh^2\theta')^{\frac{3}{2}}} \\ &= 2\int_0^{\frac{1}{2}\pi} \frac{\cos^2\phi d\phi}{\{C+S-(C-S)\sin^2\phi\}^{\frac{3}{2}}} \\ &= \frac{2}{\sqrt{k}}\int_0^{\frac{1}{2}\pi} \frac{k^2-k^2\sin^2\phi}{(1-k^2\sin^2\phi)^{\frac{3}{2}}}\,d\phi \\ &= \frac{2F}{\sqrt{k}} - \frac{2k'^2}{\sqrt{k}}\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{3}{2}}} \\ &= 2k^{-\frac{1}{2}}(F-E). \qquad\qquad (60). \end{aligned}$$

And the values of the successive Q functions can be calculated by means of the sequence equation (56).

276. At the critical circle $\eta = \infty$, and at all points on the axis of z, $\eta = 0$; and since $P = \infty$ when $\eta = \infty$, and $P = \pi$ when $\eta = 0$, the P functions are not suitable for space within a tore, but are suitable for space without the tore. On the other hand $Q_n = 0$ when $\eta = \infty$, and $= \infty$ when $\eta = 0$; hence the Q functions are suitable for space inside a tore but not for space outside.

If therefore the potential is symmetrical with respect to the axis of the tore, its proper value for points outside the tore will be

$$V = (C + \cos \xi)^{\frac{1}{2}} \Sigma_0^\infty A_n P_n (\cos n\xi + \alpha_n),$$

and for points inside

$$V' = (C + \cos \xi)^{\frac{1}{2}} \Sigma_0^\infty B_n Q_n (\cos n\xi + \alpha'_n).$$

277. A different expression for Q_n may be obtained as follows. The inverse distance of a point from the origin is

$$\frac{1}{r} = \frac{1}{a}\sqrt{\frac{C + \cos \xi}{C - \cos \xi}}.$$

Since r^{-1} is a potential function which is infinite at the origin and which vanishes at infinity, it is evident that r^{-1} can be expanded in a series such that

$$r^{-1} = a^{-1} (C + \cos \xi)^{\frac{1}{2}} \Sigma B_n Q_n \cos n\xi,$$

whence $$(C - \cos \xi)^{-\frac{1}{2}} = \Sigma B_n Q_n \cos n\xi,$$

and therefore

$$B_n Q_n = \frac{2}{\pi} \int_0^\pi \frac{\cos n\theta d\theta}{(C - \cos \theta)^{\frac{1}{2}}},$$

$$B_0 Q_0 = \frac{1}{\pi} \int_0^\pi \frac{d\theta}{(C - \cos \theta)^{\frac{1}{2}}}.$$

The quantity B_n may be some function of n, but if we substitute the above value of Q_n in the sequence equation (56), it will be found that it will be satisfied provided $B_n = A$, where A is a certain constant which is independent of n. In order to find A, we have

$$AQ_0 = \frac{1}{\pi} \int_0^\pi \frac{d\theta}{(C - \cos \theta)^{\frac{1}{2}}}$$

$$= \frac{2}{\pi\sqrt{C+1}} \int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1 - \lambda^2 \sin^2 \phi)^{\frac{1}{2}}},$$

where $\lambda^2 = 2(C+1)^{-1}$. Now $k^{-1} = C + S$, therefore

$$\frac{4k}{(1+k)^2} = \frac{4(C+S)}{(1+C+S)^2} = \frac{4}{(C+S)(C-S+1)^2}$$

$$= \frac{4}{C-S+2+C+S} = \frac{2}{C+1} = \lambda^2.$$

Therefore

$$AQ_0 = \frac{2\sqrt{2k}}{\pi(1+k)} \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\{1 - 4k(1+k^2)^{-2}\sin^2\phi\}^{\frac{1}{2}}}$$

$$= 2\pi^{-1}(2k)^{\frac{1}{2}} F(k)$$

$$= \pi^{-1} 2^{\frac{1}{2}} Q_0,$$

therefore $\quad A = \pi^{-1}\sqrt{2}.$

Similarly from the value of Q_1 it can be shown that if n is not zero, $A = 2\pi^{-1}\sqrt{2}$; therefore

$$Q_n = \int_0^\infty \frac{d\theta}{(C + S\cosh\theta)^{\frac{1}{2}(2n+1)}} = 2^{-\frac{1}{2}} \int_0^\pi \frac{\cos n\theta\, d\theta}{(C - \cos\theta)^{\frac{1}{2}}} \quad \text{......} \quad (61).$$

278. The following relations between the P and Q functions, where the accents denote differentiation with respect to η, are also useful, viz.

$$P_{n+1}Q_n - P_nQ_{n+1} = 2\pi/(2n+1) \quad \text{..............} \quad (62),$$

$$P'_nQ_n - P_nQ'_n = \pi/S \quad \text{......................} \quad (63),$$

$$P'_nQ'_{n+1} - P'_{n+1}Q'_n = \tfrac{1}{2}(2n+1)\pi \quad \text{..............} \quad (64).$$

In order to prove (62), substitute the values of P_{n+1}, Q_{n+1}, &c. from the sequence equations, and we obtain

$$(2n+1)(P_{n+1}Q_n - P_nQ_{n+1}) = (2n-1)(P_nQ_{n-1} - P_{n-1}Q_n)$$

$$= P_1Q_0 - P_0Q_1$$

$$= 4(E'F + F'E - FF')$$

$$= 2\pi.$$

The other two equations can be proved in a similar manner by means of equations (54) and (55).

279. When the motion of a liquid about a tore is symmetrical with respect to the axis of the tore, and is irrotational, we have shown that the current function $\psi = \chi' \varpi$, where χ' satisfies the equation

$$\frac{d^2\chi'}{dz^2} + \frac{d^2\chi'}{d\varpi^2} + \frac{1}{\varpi}\frac{d\chi'}{d\varpi} - \frac{\chi'}{\varpi^2} = 0,$$

whence $$\chi' = (C+c)^{\frac{1}{2}} \Sigma (A_n P_n^{\,1} + B_n Q_n^{\,1}) \cos (n\xi + \alpha_n),$$

where $c = \cos \xi$, and $P_n^{\,1}$ and $Q_n^{\,1}$ are the two solutions of the equation

$$\frac{d}{d\nu}(1-\nu^2)\frac{du}{d\nu} - \frac{u}{1-\nu^2} + (n^2 - \tfrac{1}{4})\, u = 0.$$

Differentiate (51) with respect to ν, and put

$$v = (\nu^2 - 1)^{\frac{1}{2}} \frac{d\chi}{d\nu} = S\frac{d\chi}{d\nu}.$$

Then

$$S\frac{d^2 v}{d\nu^2} + \frac{2C}{S}\frac{dv}{d\nu} - \frac{v}{S^3} - (n^2 - \tfrac{1}{4})\frac{v}{S} = 0,$$

or $$\frac{d}{d\nu}(1-\nu^2)\frac{dv}{d\nu} - \frac{v}{1-\nu^2} + (n^2 - \tfrac{1}{4})\, v = 0\,;$$

whence $$P_n^{\,1} = S\frac{dP_n}{d\nu} = \frac{dP_n}{d\eta}, \quad Q_n^{\,1} = \frac{dQ_n}{d\eta}.$$

Let us now choose two new functions U_n, V_n such that

$$U_n = SP_n^{\,1} = S\frac{dP_n}{d\eta}, \quad V_n = -SQ_n^{\,1} = -S\frac{dQ_n}{d\eta} \quad \text{......(65)},$$

and therefore

$$\frac{dU_n}{d\eta} = (n^2 - \tfrac{1}{4})\, SP_n, \quad \frac{dV_n}{d\eta} = -(n^2 - \tfrac{1}{4})\, SQ_n\,;$$

whence, remembering that $\varpi = aS(C + \cos\xi)^{-1}$, the general value of ψ is

$$\psi = (C+c)^{-\frac{1}{2}} \Sigma_0^\infty (A_n U_n + B_n V_n) \cos (n\xi + \alpha_n) \quad \text{......(66)}.$$

The function U clearly belongs to space outside the tore, and the function V to space inside; hence outside the tore the proper value of ψ is

$$\psi = (C+c)^{-\frac{1}{2}} \Sigma_0^\infty A_n U_n \cos (n\xi + \alpha_n) \quad \text{............(67)},$$

and insidc $$\psi = (C+c)^{-\frac{1}{2}} \Sigma_0^\infty B_n V_n \cos (n\xi + \alpha'_n) \quad \text{............(68)}.$$

Again,

$$V_n = -S\frac{dQ_n}{d\eta} = (n+\tfrac{1}{2})(CQ_n - Q_{n+1})$$

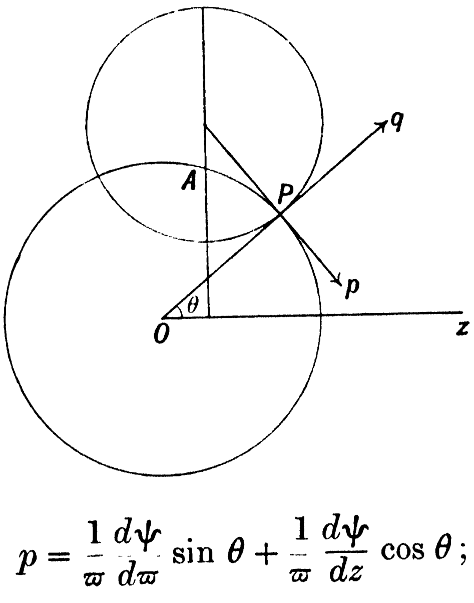
$$= \frac{2n+1}{2\sqrt{2}}\int_0^\pi \frac{C\cos n\theta - \cos(n+1)\theta}{(C-\cos\theta)^{\frac{1}{2}}}\,d\theta$$

$$= (n+\tfrac{1}{2})\,2^{-\frac{1}{2}}\int_0^\pi \{(C-\cos\theta)^{\frac{1}{2}}\cos n\theta + (C-\cos\theta)^{-\frac{1}{2}}\sin n\theta\sin\theta\}\,d\theta.$$

Integrating the last term by parts we obtain

$$V_n = -\tfrac{1}{2}(4n^2-1)\,2^{-\frac{1}{2}}\int_0^\pi (C-\cos\theta)^{\frac{1}{2}}\cos n\theta\,d\theta \quad\ldots\ldots(69).$$

280. Let p and q be the velocities perpendicular to the surfaces η and ξ, in the directions shown in the figure, then



$$p = \frac{1}{\varpi}\frac{d\psi}{d\varpi}\sin\theta + \frac{1}{\varpi}\frac{d\psi}{dz}\cos\theta\,;$$

but $\qquad \cos\theta = J\,dz/d\xi,\quad \sin\theta = J\,d\varpi/d\xi,$

therefore $\qquad p = J\varpi^{-1}\,d\psi/d\xi.$

Similarly $\qquad q = \dfrac{1}{\varpi}\dfrac{d\psi}{d\varpi}\cos\theta - \dfrac{1}{\varpi}\dfrac{d\psi}{dz}\sin\theta,$

and $\qquad \cos\theta = J\,d\varpi/d\eta,\quad \sin\theta = -J\,dz/d\eta,$

whence $\qquad q = J\varpi^{-1}\,d\psi/d\eta.$

281. We can now obtain the value of the cyclic constant; for this quantity is the circulation round any closed curve embracing the tore once. Let the curve be $\eta=\eta'$, then putting $\cos\xi=c$

$$\kappa=\int_{-\pi}^{\pi} q\,ds=\int_{-\pi}^{\pi}\frac{1}{\varpi}\frac{d\psi}{d\eta}\,d\xi$$
$$=\frac{1}{Sa}\int_{-\pi}^{\pi}(C+c)\frac{d\psi}{d\eta}\,d\xi.$$

Consider first the general term $A_nU_n\cos n\xi$ in ψ, the circulation due to this is

$$=\frac{A_n}{Sa}\int_{-\pi}^{\pi}\left\{(C+c)^{\frac{1}{2}}\frac{dU_n}{d\eta}-\frac{S}{2}(C+c)^{-\frac{1}{2}}U_n\right\}\cos n\xi\,d\xi$$
$$=2A_na^{-1}\int_0^{\pi}\left\{(C+c)^{\frac{1}{2}}(n^2-\tfrac{1}{4})P_n-\tfrac{1}{2}U_n(C+c)^{-\frac{1}{2}}\right\}\cos n\xi\,d\xi$$
$$=-A_na^{-1}(-)^n2^{\frac{1}{2}}(V_nP_n+U_nQ_n)$$
$$=SA_na^{-1}(-)^n2^{\frac{1}{2}}\left(P_n\frac{dQ_n}{d\eta}-Q_n\frac{dP_n}{d\eta}\right)$$
$$=-\pi(-)^nA_na^{-1}2^{\frac{1}{2}}.$$

Similarly the term involving V_n produces the term

$$-SB_na^{-1}2^{\frac{1}{2}}\left(Q_n\frac{dQ_n}{d\eta}-Q_n\frac{dQ_n}{d\eta}\right)=0,$$

also the terms in $\sin n\xi$ evidently disappear, whence

$$\kappa=-\pi a^{-1}2^{\frac{1}{2}}\Sigma(-)^nA_n \qquad\qquad (70).$$

282. The value of k is $\epsilon^{-\eta}$, and since η is very large in the neighbourhood of the critical circle it follows that if the cross section of the tore is small, k will be small at all points within the tore, and also at all points outside the tore which are not far from its surface.

In the hydrodynamical applications of Chapter XIV., the cross section of the tore will always be supposed to be small in comparison with its aperture, and the values of the functions will only be required at points within the tore or in its immediate neighbourhood; and it will be sufficient to employ approximate values of the functions which do not involve powers of k higher than the second.

Now if $L=\log 4/k$, and k be small,

$$F(k')=L+\tfrac{1}{4}k^2(L-1)+\tfrac{9}{64}k^4(L-\tfrac{7}{6}),$$
$$E(k')=1+\tfrac{1}{2}k^2(L-\tfrac{1}{2})+\tfrac{3}{16}k^4(L-\tfrac{7}{6}).$$

Substituting in (56), (57), (58), we obtain

$$P_0 = 2k^{\frac{1}{2}}\{L + \tfrac{1}{4}(L-1)k^2\},$$
$$P_1 = 2k^{-\frac{1}{2}}\{1 + \tfrac{1}{2}(L - \tfrac{1}{2})k^2\},$$
$$P_2 = \tfrac{4}{3}k^{-\frac{3}{2}}\{1 + \tfrac{3}{4}k^2\}.$$

The function P_n contains the factor $k^{-\frac{1}{2}(2n-1)}$ which is very large if k is small, but it will hereafter be found that $P_n(k)$ is always divided by $P_n(b)$ where b is the value of k at the surface of the tore; hence the term A_nP_n will always be of the form $A'_n(b/k)^{\frac{1}{2}(2n-1)}u_n$ where A'_n is a finite constant and u_n is a quantity of the form

$$\alpha_0 + \alpha_2k^2 + \ldots(\beta_0 + \beta_2k^2 + \beta_{2n}k^{2n} + \ldots\ldots)\,L;$$

when k is small $k^{2n}L$ is always small except when $n = 0$, also b/k can never be greater than unity, hence the preceding approximate values of P_0, P_2... may be employed.

From (56), (59) and (60) we obtain

$$Q_0 = \pi k^{\frac{1}{2}}(1 + \tfrac{1}{4}k^2),$$
$$Q_1 = \tfrac{1}{2}\pi k^{\frac{3}{2}}(1 + \tfrac{3}{8}k^2),$$
$$Q_2 = \tfrac{3}{8}\pi k^{\frac{5}{2}}(1 + \tfrac{5}{12}k^2),$$

where the series in brackets are carried to the second power only.

283. By means of these equations combined with (65) the U and V functions can be calculated, but since U_n and V_n respectively contain $k^{-\frac{1}{2}(2n+1)}$ and $\frac{1}{4}\pi k^{\frac{1}{2}(2n-1)}$ as factors, it will be more convenient to introduce two new functions R_n and T_n, such that

$$k^{-\frac{1}{2}(2n+1)}R_n = U_n,\quad \tfrac{1}{4}\pi k^{\frac{1}{2}(2n-1)}T_n = V_n \quad \ldots\ldots\ldots(71),$$

and we shall obtain

$$\left.\begin{aligned} R_0 &= -\{\tfrac{1}{2}L - 1 + \tfrac{1}{8}(L+1)k^2\} \\ R_1 &= \tfrac{1}{2}\{1 - \tfrac{3}{2}(L - \tfrac{1}{2})k^2\} \\ R_2 &= 1 - \tfrac{5}{4}k^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(72),$$

$$\left.\begin{aligned} T_0 &= 1 + \tfrac{1}{4}k^2 \\ T_1 &= \tfrac{3}{2}(1 - \tfrac{1}{8}k^2) \\ T_2 &= \tfrac{15}{8}(1 - \tfrac{1}{4}k^2) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(73),$$

where the series are carried as far as k^2. It will not be necessary to employ the functions of higher orders than R_2 and T_2, or to retain higher powers than k^2.

The general value of the current function may now be written

$$\psi = (C+c)^{-\frac{1}{2}}\Sigma_0^\infty \{A_n\,(b/k)^{\frac{1}{2}(2n+1)} R_n + B_n\,(k/b)^{\frac{1}{2}(2n-1)} T_n\} \cos(n\xi + \alpha_n) \quad \ldots\ldots(74),$$

in which form it will hereafter be employed.

EXAMPLES.

1. Apply Neumann's transformation to prove that the potential at an external point of the surface, which is the inverse of an ovary ellipsoid with respect to its focus, can be expressed by means of a series of terms of the type $cr^{-1}P_n^m(\nu)\,P_n^m(\mu)\sin(m\theta+\alpha_m)$; and at an internal point by a series of terms of the type

$$cr^{-1}Q_n^m(\nu)\,P_n^m(\mu)\sin(m\theta+\alpha_m);$$

where $\nu=\cosh\eta$, $\mu=\cos\xi$; and $z+\iota\varpi = 2c\sec^2\frac{1}{2}(\xi+\iota\eta)$.

2. Prove that

$$(1-\mu^2)^{\frac{1}{2}}(\nu^2-1)^{\frac{1}{2}}/(\mu+\nu)^3 = \tfrac{1}{2}\Sigma_1^\infty(-)^n(2n+1)\,Q'_n(\nu)\,P'_n(\mu);$$

hence show that if the surface, which is the inverse of an ovary ellipsoid with respect to its focus, be moving with velocity V parallel to its axis in an infinite liquid,

$$\psi = 8Vc^2r^{-1}\,\Sigma_1^\infty(-)^n(2n+1)\frac{Q_n^1(\gamma)}{P_n^1(\gamma)}P_n^1(\nu)\,P_n^1(\mu),$$

where γ is the value of ν at the surface.

3. Establish the following results:

(i) $$\int_0^\infty K_0(ax)\cos bx\,dx = \tfrac{1}{2}\pi(a^2+b^2)^{-\frac{1}{2}},$$

(ii) $$\int_0^\infty \epsilon^{-ax}K_0(bx)\,dx = (b^2-a^2)^{-\frac{1}{2}}\tan^{-1}(b^2-a^2)^{\frac{1}{2}}/a\,;\ b>a$$

$$= \tfrac{1}{2}(a^2-b^2)^{-\frac{1}{2}}\log\frac{a+(a^2-b^2)^{\frac{1}{2}}}{a-(a^2-b^2)^{\frac{1}{2}}}\,;\ a>b,$$

(iii) $$\int_0^\infty K_0(ax)\,J_0(bx)\,dx = (a^2+b^2)^{-\frac{1}{2}}F\{b\,(a^2+b^2)^{-\frac{1}{2}}\}.$$

4. Prove that

$$\int_0^\infty \mu^{-1}\epsilon^{-\mu z}\sin \mu c J_0(\mu\varpi)\,d\mu = \sin^{-1} 2c/(p+q),$$

where $$p^2 = z^2 + (\varpi + c)^2,\ q^2 = z^2 + (\varpi - c)^2.$$

5. By means of the definite integral

$$\int_0^\infty du \int_0^\infty \cos x \cos u^2 (x^2 - \lambda^2)\,dx,$$

prove that $$J_0(\lambda) = 2\pi^{-1}\int_0^\infty \sin(\lambda\cosh\phi)\,d\phi.$$

6. Prove that if

$$V = 2\pi^{-1}\int_0^\infty d\mu \int_0^c \epsilon^{-\mu z}\cos\lambda v \cos\mu v J_0(\mu\varpi)\,dv,$$

then $$V = J_0(\lambda\varpi), \text{ when } z = 0 \text{ and } \varpi < c,$$

and that $$dV/dz = 0, \text{ when } z = 0 \text{ and } \varpi > c,$$

7. Prove that if

$$V = 2\pi^{-1}\int_0^\infty d\mu \int_0^c \epsilon^{-\mu z}\sin\lambda v \sin\mu v J_1(\mu\varpi)\,dv,$$

then $$V = J_1(\lambda\varpi), \text{ when } z = 0, \text{ and } \varpi < c,$$

and that $$dV/dz = 0, \text{ when } z = 0 \text{ and } \varpi > c.$$

8. Prove that

$$\int_0^\infty \epsilon^{-ax}\{J_0(bx)\}^2\,dx = 2\pi^{-1}(a^2+4b^2)^{-\frac{1}{2}} F\{2b(a^2+4b^2)^{-\frac{1}{2}}\}.$$

CHAPTER XIII.

RECTILINEAR VORTICES.

284. THE general theory of vortex motion has been discussed in Chapter IV.; and we shall now consider the special case in which all the vortex lines are parallel to the axis of z. We shall also include the case in which cylindrical masses of rotationally moving liquid composed of such vortex lines are surrounded by irrotationally moving liquid. If the whole liquid is supposed to extend to infinity in the positive and negative directions of the axis of z, and the boundaries of the liquid consist of cylinders whose generating lines are parallel to this axis, the problem will evidently be one of two-dimensional motion, and the solution will apply to any limited portion of the liquid bounded by two fixed planes perpendicular to the axis of z.

Since the motion is in two dimensions,

$$w = 0,\ du/dz = 0,\ dv/dz = 0,\ \xi = 0,\ \eta = 0,$$

and

$$\frac{dv}{dx} - \frac{du}{dy} = 2\zeta \quad\text{.........................(1)};$$

also $\partial\zeta/\partial t = 0$, and therefore ζ remains constant for each particular element of liquid. If ψ be the current function, $u = d\psi/dy$, $v = -\,d\psi/dx$; whence, substituting in (1), we obtain

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} + 2\zeta = 0 \quad\text{....................(2).}$$

This equation must be satisfied at every point of the liquid where vortex motion exists. At every point of the irrotationally moving liquid which surrounds the vortices $\zeta = 0$, and therefore

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} = 0 \quad\text{......................(3).}$$

Equations (2) and (3) show that ψ is the potential of indefinitely long cylinders composed of attracting matter of density $\zeta/2\pi$, which occupy the same positions as the vortices.

285. Let us now suppose that a single rectilinear vortex, whose cross section is a circle of radius a, exists in an infinite liquid. In order that the cross section may remain circular, it is necessary that ζ and ψ should be functions of r alone. The conditions of steady motion § 38 (37) require that ζ should be equal to an arbitrary function of ψ, which for the present we shall suppose to be equal to a constant.

Equations (2) and (3) now become

$$\frac{d^2\psi_1}{dr^2}+\frac{1}{r}\frac{d\psi_1}{dr}+2\zeta=0 \quad\text{.................(4)},$$

which gives the values of ψ inside the vortex, and

$$\frac{d^2\psi_2}{dr^2}+\frac{1}{r}\frac{d\psi_2}{dr}=0 \quad\text{....................(5)},$$

which gives the value outside.

The complete integrals of (4) and (5) are

$$\psi_1=A\log r+B-\tfrac{1}{2}\zeta r^2$$

and $$\psi_2=C\log r+D.$$

Now ψ_1 must not be infinite when $r=0$, and therefore $A=0$; also at the boundary of the vortex, where $r=a$,

$$\psi_1=\psi_2, \qquad d\psi_1/dr=d\psi_2/dr;$$

whence $$B-\tfrac{1}{2}\zeta a^2=C\log a+D$$

$$-\zeta a^2=C,$$

and therefore $$C=-\zeta a^2=-\zeta\sigma/\pi=-m,$$

where σ is the area of the cross section, and πm is the *strength* of the vortex. The constant D contributes nothing to the velocity, and may therefore be omitted, whence

$$\psi_1=\tfrac{1}{2}\zeta(a^2-r^2)-m\log a \quad\text{.................(6)},$$

$$\psi_2=-m\log r \quad\text{..............................(7)}.$$

Now $-d\psi/dr$ is the velocity perpendicular to r, whence inside the vortex

$$-d\psi_1/dr=\zeta r \quad\text{......................(8)},$$

which vanishes when $r=0$, and outside

$$-d\psi_2/dr=m/r \quad\text{......................(9)}.$$

Hence a single vortex whose cross section is circular, if existing in an infinite liquid will remain at rest, and will rotate as a rigid body. It will also produce at every point of the irrotationally moving liquid with which it is surrounded, a velocity which is perpendicular to the line joining that point with the centre of its cross section, and which is inversely proportional to the distance of that point from the centre.

If ϕ be the velocity potential outside the vortex

$$\begin{aligned}\phi &= -\tfrac{1}{2} m\iota \log (x+\iota y)/(x-\iota y)\\ &= m \tan^{-1} y/x \quad\text{......................... ......(10)},\end{aligned}$$

whence ϕ is a monocyclic function whose cyclic constant is $2\pi m$; and therefore if κ be the circulation due to the vortex, its strength is equal to $\frac{1}{2}\kappa$.

286. If other vortices exist in the liquid, or if the liquid instead of extending to infinity is bounded by fixed or moving surfaces, the cross section, if of finite area, will experience a deformation, and the preceding expressions for ϕ and ψ will not continue to hold; but we shall hereafter show that if the cross section is small, this deformation may be neglected, and (6), (7) and (10) will give the values of ϕ and ψ so far as this particular vortex is concerned. Also since every vortex of finite cross section may be divided into elementary vortex filaments, the value of ψ at any point (x, y) for any number of vortices will be

$$\psi = -2^{-1}\pi^{-1}\iint \zeta \log \{(x-x')^2+(y-y')^2\}\, dx'dy' \quad\text{......(11)},$$

where the integration extends over the cross sections of all the vortices.

It therefore follows that the component velocities due to any number of vortices will be determined by the superposition of the velocities due to each, and will be given by the equations

$$u = -\Sigma m (y-y_1)/R^2, \quad v = \Sigma m (x-x_1)/R^2,$$

where $R^2 = (x-x_1)^2 + (y-y_1)^2$, and (x_1, y_1) are the coordinates of any of the vortices. Now if (u, v) be the component velocities at any point of one of the vortices the expressions $\Sigma\,(mu)$ and $\Sigma\,(mv)$, where the summations extend throughout the vortices, vanish; for they each consist of pairs of terms of the forms $m_1 m_2 (x_1 - x_2)/R^2$ and $m_1 m_2 (x_2 - x_1)/R^2$. Hence if m be regarded as the mass of a distribution of matter, the centre of inertia of this mass remains stationary throughout the motion.

287. Let us now suppose that in the irrotationally moving liquid which surrounds a vortex whose cross section is circular, the circulation is different from that which is due to the vortex, and consequently the tangential velocities at the common surface of the vortex and the surrounding liquid are different on either sides of this surface. This surface will therefore be a surface of discontinuity which possesses the properties of a vortex sheet. We shall also for greater generality suppose that the density of the liquid forming the vortex is different from that surrounding it.

Let σ be the density of the vortex, κ' the circulation due to it; ρ the density of the outside liquid, κ its circulation; also let ψ', ψ be the current functions inside and outside the vortex.

Then $$\psi' = -\tfrac{1}{2}\zeta r^2 + \text{const.},$$

and $$\kappa' = -a\int_0^{2\pi} (d\psi'/dr)_a\, d\theta = 2\pi\zeta a^2.$$

Therefore $$\psi' = -\kappa' r^2/4\pi a^2 + \text{const.},$$

$$\psi = -\kappa/2\pi\,.\log r + \text{const.}$$

Let p', p be the pressures in the vortex and the surrounding liquid, then

$$\frac{1}{\sigma}\frac{dp'}{dr} = \frac{v^2}{r} = \frac{\kappa'^2 r}{4\pi^2 a^4}.$$

Therefore $$\frac{p'}{\sigma} = \frac{\kappa'^2 r^2}{8\pi^2 a^4} + \frac{P}{\sigma},$$

also $$\frac{p}{\rho} = \frac{\Pi}{\rho} - \frac{\kappa^2}{8\pi^2 r^2},$$

where P is the pressure at the centre of the cross section of the vortex, and Π is the pressure at infinity. At the surface of separation $p = p'$, whence

$$P = \Pi - (\kappa^2\rho + \kappa'^2\sigma)/8\pi^2 a^2.$$

Hence if $$\Pi < (\kappa^2\rho + \kappa'^2\sigma)/8\pi a^2,$$

p' will become negative for some value of $r < a$, which shows that a cylindrical hollow will exist in the vortex, which is concentric with its outer boundary.

The case of $\sigma = 0$ is that of a cylindrical hollow surrounded by liquid in a state of cylic irrotational motion. The condition for the existence of such a hollow is that $p = 0$ when $r = a$, hence

$$\Pi = \kappa^2\rho/8\pi^2 a^2.$$

288. We must now investigate the stability of the preceding case of steady motion.

Let us suppose that a small disturbance is communicated to the liquid; the equation of the common surface of separation may be taken to be of the form

$$F = a + \alpha \cos n\theta + \beta \sin n\theta - r = 0 \quad\text{............(12)},$$

where n is any positive integer, and α, β are functions of the time which in the beginning of the disturbed motion are small quantities, whose squares and products may be neglected.

Let the current functions be

$$\psi = -\kappa/2\pi \,.\, \log r + (A \cos n\theta + B \sin n\theta)\,(a/r)^n \quad\text{...(13)}$$

outside the vortex, and

$$\psi' = -\kappa' r^2/4\pi a^2 + (C \cos n\theta + D \sin n\theta)\,(r/a)^n \quad\text{......(14)}$$

inside the vortex. The boundary condition is

$$\frac{dF}{dt} + \frac{1}{r}\frac{dF}{dr}\frac{d\psi}{d\theta} - \frac{1}{r}\frac{dF}{d\theta}\frac{d\psi}{dr} = 0 \quad\text{............(15)}.$$

Substituting the value of F from (12) we obtain

$$\dot\alpha \cos n\theta + \dot\beta \sin n\theta - \frac{1}{r}\frac{d\psi}{d\theta} + \frac{n}{r}\frac{d\psi}{dr}(\alpha \sin n\theta - \beta \cos n\theta) = 0.$$

If U be the tangential velocity of the surrounding liquid at the surface of separation in steady motion, we may in the small terms put $d\psi/dr = -U$, whence

$$\dot\alpha \cos n\theta + \dot\beta \sin n\theta + na^{-1}(A \sin n\theta - B \cos n\theta)$$
$$- nUa^{-1}(\alpha \sin n\theta - \beta \cos n\theta) = 0.$$

Equating the coefficients of $\sin n\theta$, $\cos n\theta$ to zero, we obtain

$$\left.\begin{aligned} A &= -a\dot\beta/n + U\alpha \\ B &= a\dot\alpha/n + U\beta \end{aligned}\right\} \quad\text{....................(16)}.$$

Similarly if U' be the tangential velocity of the vortex at the surface of separation in steady motion, we shall obtain

$$\left.\begin{aligned} C &= -a\dot\beta/n + U'\alpha \\ D &= a\dot\alpha/n + U'\beta \end{aligned}\right\} \quad\text{....................(17)}.$$

Since the disturbed motion will necessarily be irrotational it will have a velocity potential, and by employing the method of conjugate functions it can easily be shown that

$$\phi = (A \sin n\theta - B \cos n\theta)\,(a/r)^n,$$

$$\phi' = -(C \sin n\theta - D \cos n\theta)\,(r/a)^n.$$

If δp, $\delta p'$ be the increments of the pressure due to the disturbed motion just outside and just inside the vortex, we must have

$$\frac{\delta p}{\rho} = -\dot{\phi} - \tfrac{1}{2}\left(\frac{d\psi}{dr}\right)^2 + \tfrac{1}{2}U^2$$

$$= -(\dot{A}\sin n\theta - \dot{B}\cos n\theta) + U^2 a^{-1}(\alpha\cos n\theta + \beta\sin n\theta)$$

$$- nUa^{-1}(A\cos n\theta + B\sin n\theta)$$

$$= (-\dot{A} + U^2\beta/a - nUB/a)\sin n\theta + (\dot{B} + U^2\alpha/a - nUA/a)\cos n\theta$$

$$= \{a\ddot{\beta}/n - 2U\dot{\alpha} - U^2\beta(n-1)/a\}\sin n\theta$$

$$+ \{a\ddot{\alpha}/n + 2U\dot{\beta} - U^2\alpha(n-1)/a\}\cos n\theta \text{.........(18)},$$

by (16). From the general equations of motion we have

$$-\frac{1}{\sigma}\frac{dp'}{dx} = \frac{du}{dt} + \tfrac{1}{2}\frac{dq^2}{dx} - 2\zeta v,$$

$$-\frac{1}{\sigma}\frac{dp'}{dy} = \frac{dv}{dt} + \tfrac{1}{2}\frac{dq^2}{dy} + 2\zeta u,$$

whence $$-p'/\sigma = \dot{\phi}' + \tfrac{1}{2}q^2 + 2\zeta\psi'.$$

Hence

$$\frac{\delta p'}{\sigma} = -\dot{\phi}' - \tfrac{1}{2}\left(\frac{d\psi'}{dr}\right)^2 + \tfrac{1}{2}U'^2 - \frac{2U'}{a}\left(\psi' - \frac{\kappa'^2}{4\pi}\right)$$

$$= (\dot{C}\sin n\theta - \dot{D}\cos n\theta) - U'^2a^{-1}(\alpha\cos n\theta + \beta\sin n\theta)$$

$$+ U'na^{-1}(C\cos n\theta + D\sin n\theta) + 2U'^2a^{-1}(\alpha\cos n\theta + \beta\sin n\theta)$$

$$- 2U'a^{-1}(C\cos n\theta + D\sin n\theta)$$

$$= \{\dot{C} + U'^2\beta/a - U'D(n-2)/a\}\sin n\theta$$

$$+ \{-\dot{D} + U'^2\alpha/a - U'C(n-2)/a\}\cos n\theta$$

$$= \{-a\ddot{\beta}/n + 2U'(n-1)\dot{\alpha}/n + U'^2\beta(n-1)/a\}\sin n\theta$$

$$+ \{-a\ddot{\alpha}/n - 2U'(n-1)\dot{\beta}/n + U'\alpha(n-1)/a\}\cos n\theta \text{.........(19)},$$

by (17). In (18) and (19) write α and β for $a\alpha$ and $a\beta$, and w and v for U/a and U'/a; since $\delta p = \delta p'$, we obtain by equating the coefficients of $\sin n\theta$, $\cos n\theta$ in the expressions for δp, $\delta p'$ given by (18) and (19),

$$\left.\begin{aligned}\ddot{\alpha}(1+\sigma/\rho) + 2\dot{\beta}\{nw + v(n-1)\sigma/\rho\} - n(n-1)\alpha\{w^2+v^2\sigma/\rho\} &= 0\\ \ddot{\beta}(1+\sigma/\rho) - 2\dot{\alpha}\{nw + v(n-1)\sigma/\rho\} - n(n-1)\beta\{w^2+v^2\sigma/\rho\} &= 0\end{aligned}\right\}\text{(20)}.$$

To solve these equations put $\alpha = L\cos\lambda t$, $\beta = L\sin\lambda t$, also let $k = \sigma/\rho$, and we obtain

$$\lambda^2(1+k) - 2\lambda\{nw + k(n-1)v\} + n(n-1)(w^2 + kv^2) = 0\text{...(21)}.$$

In order that the steady motion should be stable, it is necessary that both roots of this quadratic should be real.

Case I. Suppose that there is no core, but simply a cylindrical hollow round which circulation takes place. Here $\sigma = 0$, $\delta p = 0$, whence from (18) or (20) we obtain

$$\lambda^2 - 2\lambda n w + n(n-1) w^2 = 0,$$

the roots of which are $(n \pm \sqrt{n}) w$. Hence the steady motion is stable, and the disturbance consists of two trains of waves travelling round the ring in the same direction.

Case II. Let the vortex be of the same density as the surrounding liquid, and let there be no slipping at the surface of separation. Here $w = v = \zeta$, $\rho = \sigma$, $k = 1$ and (21) becomes

$$\lambda^2 - \lambda\zeta(2n-1) + n(n-1)\zeta^2 = 0,$$

the roots of which are $n\zeta$ and $(n-1)\zeta$. Hence the steady motion is stable. It might at first sight appear that the disturbance consists of *two* trains of waves whose periods are $2\pi/n\zeta$, and $2\pi/(n-1)\zeta$ respectively; but in order to solve this case it is unnecessary to take into account the pressure condition, since the two values of ψ at the surface of separation must differ by a constant quantity, which together with the condition of no slipping and the boundary condition (15) are sufficient to determine the disturbed motion. It will thus be found that the equations of motion become

$$\ddot{\alpha} + (n-1)^2 \zeta^2 \alpha, \quad \ddot{\beta} + (n-1)^2 \zeta^2 \beta = 0,$$

and therefore the solution $\lambda = n\zeta$ of (21) must be rejected, and the disturbance consists of a train of waves travelling round the cylinder whose period[1] is $2\pi/(n-1)\zeta$.

Case III. In the general case the condition that the roots of (21) should be real is that

$$\{nw + k(n-1)v\}^2 - n(n-1)(k+1)(w^2 + kv^2) > 0,$$

or

$$2kn(n-1)wv - n\{(n-1)k - 1\}w^2 - k(n-1)(n+k)v^2 > 0 \ldots (22).$$

If $w = v$ the condition becomes

$$n - k^2(n-1) > 1,$$

[1] Sir W. Thomson, "On the Vibrations of a Columnar Vortex," *Phil. Mag.* Sep. 1880. J. J. Thomson, *Motion of Vortex Rings*, p. 74.

which requires that $k < 1$. The steady motion will therefore be unstable if the density of the vortex is greater than that surrounding liquid.

Let $w/v = 1 + q$; then (22) becomes on dividing by n^2,

$$\frac{1-k^2}{n} + \frac{k^2}{n^2} + \frac{2q}{n} - q^2\left\{\left(1 - \frac{1}{n}\right) k - \frac{1}{n}\right\} > 0.$$

If q is not zero, it is always possible by taking n large enough, to make the left-hand side negative; hence the motion is unstable, unless $w = v$, and $\sigma \leqq \rho$.

When w and v are unequal, the common surface of separation is a surface of discontinuity which has the properties of a vortex sheet, and the preceding investigation confirms Sir W. Thomson's statement that discontinuous motion is unstable.

289. Kirchhoff[1] has shown that it is possible for a vortex whose cross section is an invariable ellipse, and whose vorticity at every point is constant, to rotate in a state of steady motion in an infinite liquid, provided a certain relation exists between the vorticity and the angular velocity of the axes of the cross section.

The current function is evidently equal to the potential of an elliptic cylinder of density $\zeta/2\pi$. Let a and b be the semi-axes of the cross section, and let the value of ψ inside the vortex be

$$\psi' = D - \zeta(Ax^2 + By^2)/(A + B).$$

Let $x = c\cosh\eta\cos\xi$, $y = c\sinh\eta\sin\xi$, where $c = (a^2 - b^2)^{\frac{1}{2}}$, and let $\eta = \beta$ at the surface; the value of ψ becomes

$$\psi' = D - \zeta c^2(A\cosh^2\eta\cos^2\xi + B\sinh^2\eta\sin^2\xi)/(A + B).$$

Also let the value of ψ outside the vortex be

$$\psi = A'\epsilon^{-2\eta}\cos 2\xi + D\eta/\beta.$$

When $\eta = \beta$, we must have

$$\psi - \psi' = \text{const.},\quad d\psi/d\eta = d\psi'/d\eta.$$

Therefore $\quad A'\epsilon^{-2\beta} = -\tfrac{1}{2}\zeta c^2(A\cosh^2\beta - B\sinh^2\beta)/(A + B)$

and $\quad A'\epsilon^{-2\beta} = \tfrac{1}{2}\zeta c^2(A - B)\sinh\beta\cosh\beta/(A + B).$

Whence $\quad A'(a-b)^2 = -\dfrac{\zeta c^2(Aa^2 - Bb^2)}{2(A+B)} = \dfrac{\zeta c^2(A-B)ab}{2(A+B)}$

Therefore $Aa = Bb$ and

$$\psi' = D - \zeta(bx^2 + ay^2)/(a + b).$$

[1] *Vorles. über Math. Phy.* p. 261, see also Hill, *Phil. Trans.* 1884, p. 363.

Let ω be the angular velocity of the axes; u, v the velocities of the liquid parallel to them, then

$$\dot{x} - y\omega = u = d\psi'/dy = -2a\zeta y/(a+b),$$

$$\dot{y} + x\omega = v = -d\psi'/dx = 2b\zeta x/(a+b).$$

The boundary condition is

$$\dot{x}\frac{dF}{dx} + \dot{y}\frac{dF}{dy} = 0,$$

where $F = (x/a)^2 + (y/b)^2 - 1 = 0$. Whence

$$\left(\omega - \frac{2a\zeta}{a+b}\right)\frac{1}{a^2} + \left(\frac{2b\zeta}{a+b} - \omega\right)\frac{1}{b^2} = 0,$$

therefore $$\omega = 2ab\zeta/(a+b)^2.$$

We therefore obtain

$$\dot{x} = -a\omega y/b, \quad \dot{y} = b\omega x/a,$$

the integrals of which are

$$x = La\cos(\omega t + \alpha), \quad y = Lb\sin(\omega t + \alpha),$$

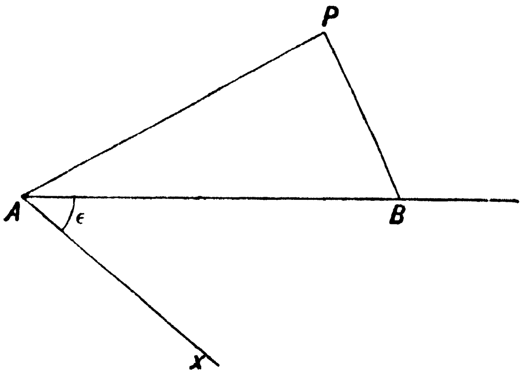
where L and α are the constants of integration. Whence the path of every particle relative to the boundary, is a similar ellipse.

290. We have shown that the effect of a cylindrical vortex column of small cross section is to produce at every point P external to it, a velocity whose magnitude is equal to m/r and whose direction is perpendicular to that of r, where r is the distance of P from vortex. If therefore more than one vortex exists in the liquid, the effect of any one of the vortices upon the others will be to produce a motion of translation combined with a deformation of their cross sections. The mathematical difficulties of solving this problem when the initial distribution of the vortices and the initial forms of their cross sections are given, are very great; and it seems impossible in the present state of analysis to do more than obtain an approximate solution in certain cases. We shall now show that when there are two rectilinear vortices in a liquid, the linear dimensions of whose cross sections are small in comparison with the shortest distance between them, the cross sections will remain approximately circular[1]; from which it is inferred that a similar result holds good in the case of any number of vortices.

[1] J. J. Thomson, *Motion of Vortex Rings*, p. 74.

Hence it follows that if any number of vortices of small cross sections are moving in the liquid, and the vortices never get very close to one another, we may neglect the effects produced by the deformations of their cross sections, which may therefore be regarded as approximately circular.

291. Let A and B be the centres of the two vortices at time t; ϵ the angle which the line joining their centres makes with some line fixed in space; also let (r, θ) be the coordinates of any point referred to the centre of A, and Ax as initial line, and let ζ be the vorticity of A.



Let the equation of the cross section of A be

$$r = a + \Sigma\,(\alpha_n \cos n\theta + \beta_n \sin n\theta)\qquad\qquad(23),$$

and let the values of the current functions outside and inside A be

$$\psi = C - \zeta a^2 \log r + \Sigma\,(A_n \cos n\theta + B_n \sin n\theta)\,(a/r)^n,$$

and $$\psi_1 = C_1 - \tfrac{1}{2}\zeta r^2 + \Sigma\,(C_n \cos n\theta + D_n \sin n\theta)\,(r/a)^n.$$

Since we suppose that α, β, A, B, C, D are all small quantities, whose squares and products are to be neglected, it follows that the condition that the values of ψ and ψ_1 should differ by a constant quantity at the surface of the vortex is that

$$A_n = C_n, \quad B_n = D_n.$$

Also since we assume that there is no slipping at the surface of A, the values of $d\psi/dr$ and $d\psi_1/dr$ must be equal at the surface; this condition gives

$$A_n = a\zeta\alpha_n/n, \quad B_n = a\zeta\beta_n/n,$$

and therefore the value of ψ is

$$\psi = C - \zeta a^2 \log r + a\zeta\,\Sigma\,(\alpha_n \cos n\theta + \beta_n \sin n\theta)\,a^n/nr^n.$$

Let us now denote corresponding quantities which refer to the other vortex B by accented letters, and we have

$$\psi' = C' - \zeta' b^2 \log r' + b\zeta' \Sigma (\alpha'_n \cos n\theta + \beta'_n \sin n\theta)\, b^n/nr'^n,$$

where b is the mean radius of the section of B.

If R, Θ be the velocities of any point on the surface of A, relative to its centre, the boundary condition is

$$\frac{dF}{dt} + \frac{R}{r}\frac{dF}{dr} + \frac{\Theta}{r}\frac{dF}{d\theta} = 0,$$

where the value of F is given by (23); whence

$$\dot{\alpha}_n \cos n\theta + \dot{\beta}_n \sin n\theta - \left\{\frac{1}{a}\frac{d(\psi+\psi')}{d\theta} + \frac{\zeta' b^2}{c}\sin(\theta-\epsilon)\right\}$$

$$- n(\alpha_n \sin n\theta - \beta_n \cos n\theta)\left\{-\frac{d(\psi+\psi')}{dr} + \frac{\zeta' b^2}{c}\cos(\theta-\epsilon)\right\} = 0, \quad (24).$$

Now $$\frac{d\psi}{d\theta} = -a\zeta\Sigma(\alpha_n \sin n\theta - \beta_n \cos n\theta),$$

also $$\frac{d\psi'}{d\theta} = -\zeta' b^2 \frac{d}{d\theta}\log r',$$

the portion involving the series being neglected, since it involves terms of the order ab/c &c., which are of a higher order than the first. But if $c > r$,

$$\log r' = \tfrac{1}{2}\log\{r^2 + c^2 - 2rc\cos(\theta-\epsilon)\}$$
$$= \log c - r/c \,.\cos(\theta-\epsilon) - \tfrac{1}{2} r^2/c^2 \,.\cos 2(\theta-\epsilon) - \&c.,$$

therefore $$\frac{d\psi'}{d\theta} = -\zeta' b^2 \{a/c \,.\sin(\theta-\epsilon) + a^2/c^2 .\sin 2(\theta-\epsilon) + \&c.\}.$$

Also $$\frac{d\psi}{dr} = -\zeta a,$$

and $$\frac{d\psi'}{dr} = \zeta' b^2 \{c^{-1}\cos(\theta-\epsilon) + ac^{-2}\cos 2(\theta-\epsilon) + \&c.\},$$

whence (24) becomes

$$\dot{\alpha}_n \cos n\theta + \dot{\beta}_n \sin n\theta + \zeta\Sigma(\alpha_n \sin n\theta - \beta_n \cos n\theta)$$
$$+ \zeta' b^2 ac^{-2} \sin 2(\theta-\epsilon) - n\zeta(\alpha_n \sin n\theta - \beta_n \cos n\theta) = 0.$$

Equating the coefficients of $\sin\theta$, $\cos\theta$, we obtain

$$\dot{\alpha}_1 = 0, \quad \dot{\beta}_1 = 0,$$

and since α_1, β_1 are initially zero, they will remain so during the whole motion; hence the centre of inertia of either vortex column

is undisturbed. Equating the coefficients of $\sin 2\theta$, $\cos 2\theta$ we obtain

$$\dot{\alpha}_2 + \zeta\beta_2 = \zeta' ab^2c^{-2} \sin 2\epsilon,$$
$$\dot{\beta}_2 - \zeta\alpha_2 = -\zeta' ab^2c^{-2} \cos 2\epsilon.$$

Since the centre of inertia of neither vortex column is disturbed, their common centre of inertia will remain at rest, and the two vortices will revolve around it with angular velocity n, where $n = (\zeta a^2 + \zeta' b^2)/c^2$; whence $\epsilon = nt$, and our equations become

$$\dot{\alpha}_2 + \zeta\beta_2 = \zeta' ab^2c^{-2} \sin 2nt,$$
$$\dot{\beta}_2 - \zeta\alpha_2 = -\zeta' ab^2c^{-2} \cos 2nt,$$

therefore $$\ddot{\alpha}_2 + \zeta^2\alpha_2 = \zeta' ab^2c^{-2}(2n + \zeta) \cos 2nt,$$

whence $$\alpha_2 = A \cos(\zeta t + \beta) + \frac{\zeta' ab^2 (2n + \zeta) \cos 2nt}{c^2(\zeta^2 - 4n^2)},$$

with a similar equation for β_2. Let the initial values of α_2, β_2, $\dot{\alpha}_2$, be zero, and we obtain

$$\alpha_2 = \frac{\zeta' ab^2}{c^2(\zeta - 2n)}(\cos 2nt - \cos \zeta t),$$

$$\beta_2 = \frac{\zeta' ab^2}{c^2(\zeta - 2n)}(\sin 2nt - \sin \zeta t).$$

Hence the cross section at any instant is an ellipse whose axes are functions of the time, and which vibrates about the circular form. The vibration has two periods, a long one π/n and a short one $2\pi/\zeta$.

292. We shall pass on to consider the motion of a number of vortices of small and approximately circular cross sections.

Since we neglect deformations of the cross sections, the current function due to each vortex will be $-m \log r$, and the velocity due to it at any point P will be m/r, and will be perpendicular to the line joining P with the vortex. Hence if two vortices of equal strengths m exist in a liquid, each vortex will describe a circle whose centre is the middle point of the line joining them, with velocity $m/2c$, where $2c$ is the distance between them; and therefore each vortex will move as if there existed a stress in the nature of a tension between them, of magnitude $m^2/4c^3$.[1]

To find the stream lines relative to the line joining the vortices,

[1] Greenhill, "Plane Vortex Motion," *Quart. Journ.* vol. xv. p. 20.

take moving axes, in which the axis of x coincides with the above-mentioned line; then

$$\psi = -\tfrac{1}{2}m \log \{y^2 + (x-c)^2\}\{y^2 + (x+c)^2\}.$$

Also
$$\dot{x} - \omega y = u = d\psi/dy,$$
$$\dot{y} + \omega x = v = -d\psi/dx,$$

where $\omega = m/2c^2$. Let

$$\chi = \psi + \tfrac{1}{2}\omega(x^2 + y^2),$$

therefore
$$\dot{x} = d\chi/dy\,,\ \dot{y} = -d\chi/dx.$$

Multiplying by $\dot{y}, \dot{x}$ respectively, subtracting and integrating, we obtain

$$\chi = \text{const.} = A,$$

whence the equation of the relative stream lines is

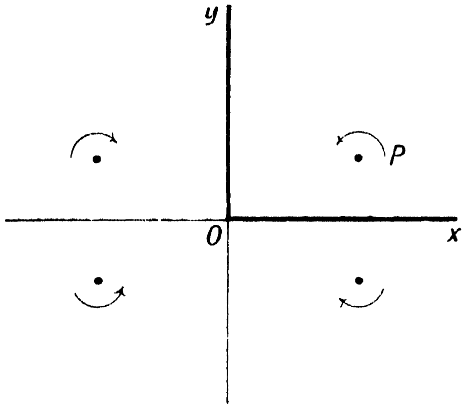
$$\tfrac{1}{2}\omega(x^2 + y^2) - \tfrac{1}{2}m \log \{y^2 + (x-c)^2\}\{y^2 + (x+c)^2\} = A.$$

293. If two opposite vortices of strengths πm and $-\pi m$ are present in the liquid, the vortices will move perpendicularly to the line joining them with velocity $m/2c$, where $2c$ is the distance between them.

In this case there is evidently no flux across the plane which bisects the line joining the vortices, and which is perpendicular to it; we may therefore remove one of the vortices and substitute this plane for it. Hence a vortex in a liquid which is bounded by a fixed plane will move parallel to the plane, and the motion of the liquid will be the same as would be caused by the original vortex, together with another vortex of equal and opposite strength, which is at an equal distance and on the opposite side of the plane.

This vortex is evidently the image of the original vortex, and we may therefore apply the theory of images in considering the motion of vortices in a liquid bounded by planes.

294. If there is a vortex at the point (x, y) moving in a square corner bounded by the planes Ox, Oy, the images will consist of two negative vortices at the points $(-x, y)$, $(x, -y)$, and a positive vortex at the point $(-x, -y)$; for if these vortices be substituted for the planes, their combined effect will be to cause no flux across them.



Since the vortex is incapable of producing any motion of translation upon itself, its motion will be due solely to that produced by the combined effect of its images; whence,

$$\dot{x} = \frac{m}{2y} - \frac{my}{2(x^2+y^2)} = \frac{mx^2}{2y(x^2+y^2)},$$

$$\dot{y} = -\frac{m}{2x} + \frac{mx}{2(x^2+y^2)} = -\frac{my^2}{2x(x^2+y^2)};$$

therefore $$\dot{x}/x^3 + \dot{y}/y^3 = 0$$

whence $$x^{-2} + y^{-2} = a^{-2}$$

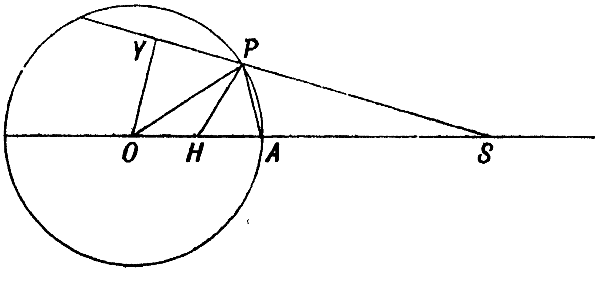
or $$r \sin 2\theta = 2a.$$

This is the equation of a Cotes' Spiral, which is the curve described by the vortices: also since

$$x\dot{y} - \dot{x}y = -\tfrac{1}{2}m$$

the vortex describes the spiral in exactly the same way as a particle would describe it, if repelled from the origin with a force $3m^2/4r^3$.

295. The method of images may also be applied to determine the current function due to a vortex in a liquid, which is bounded externally or internally by a circular cylinder.



Let H be the vortex, a the radius of the cylinder, $OH = c$; and

let S be a point such that $OS = f = a^2/c$, then the triangles SOP and POH are similar, therefore

$$SPO = OHP,$$

$$OPH = OSP,$$

also $$OSP + SPA = OAP = OPA$$

$$= OPH + HPA,$$

therefore $$SPA = HPA.$$

Let us place another vortex of equal and opposite strength at S, then the velocity along OP due to the two vortices is

$$u = -\frac{m}{HP}\sin HPO + \frac{m}{SP}\sin SPO.$$

But $$\frac{\sin HPO}{\sin SPO} = \frac{\sin HPO}{\sin OHP}$$

$$= OH/a$$

$$= HP/SP,$$

hence $u = 0$ and there is no flux across the cylinder.

Hence the image of a vortex inside a cylinder, is another vortex of equal and opposite strength situated on the line joining the vortex with the centre of the cylinder, and at a distance a^2/c from the centre, and the vortex and its image will describe circles about the centre with a velocity

$$m/SH = mc/(a^2 - c^2).$$

The velocities of the vortex and its image are equal, but their angular velocities about the axis of the cylinder will be different; hence the motion of the liquid inside the cylinder and the motion of the liquid outside the cylinder are independent, and the vortex and its image will not remain on the same radial plane in the subsequent motion. Hence the motions of the liquid inside and outside the cylinder do not correspond, as is the case with plane boundaries, except at the instant when the vortex and its image are on the same radial plane.

The current function of the liquid at a point (r, θ) within the cylinder is

$$\psi = -m\log SP/HP$$

$$= -\tfrac{1}{2}m\log\frac{r^2 + c^2 - 2rc\cos\theta}{r^2 + f^2 - 2rf\cos\theta}.$$

296. The current function due to a vortex situated between two parallel planes, can be obtained by finding the current function due to the two infinite trails of images, exactly in the same manner as the velocity potential due to a source under the same circumstances, was found in § 57.

Let the origin be midway between the two planes, and the axis of x perpendicular to them, then as in § 57,

$$\psi = f(x-x', \ y-y') - f(x+x'+2a, \ y-y') + C$$

where

$$f(x, y) = -\tfrac{1}{2}m \log \Pi_{\infty}^{\infty} \{(x+4na)^2 + y^2\}.$$

Now if we omit constant terms, we have shown that

$$f(x, y) = -\tfrac{1}{2}m \log (\cosh \pi y/2a - \cos \pi x/2a),$$

therefore

$$\psi = -\tfrac{1}{2}m \log \frac{\cosh \pi (y-y')/2a - \cos \pi (x-x')/2a}{\cosh \pi (y-y')/2a + \cos \pi (x+x')/2a}.$$

297. Let us now transform the preceding expression by putting $(x+\iota y)/a = (x_1+\iota y_1)^{\frac{1}{2}}/c^{\frac{1}{2}}$, $(x'+\iota y')/a = (x'_1+\iota y'_1)^{\frac{1}{2}}/c^{\frac{1}{2}}$.

The portions of the lines $x = \pm a$ which lie on the positive side of the axis of x, evidently become transformed into a parabola, and the portion of space lying on the positive side of the axis of x, which is bounded by these lines and the portion of the axis of x which is intercepted between them, becomes transformed into the space inside the parabola; whilst the portion of space bounded by these lines which lies on the negative side of the axis of x altogether disappears. Also the portion of the axis of x which is intercepted between the lines $x = \pm a$, transforms into a double line joining the focus of the parabola with its vertex. Now if we were to transform the preceding expressions for ψ as above mentioned, it would be found that the velocity at points on the line joining the focus of the parabola with its vertex would be discontinuous; but if we place another vortex of equal strength at the point $-x', -y'$, and add the results, the velocity in the transformed expression will be continuous along this line. We thus obtain the current function due to a vortex in the parabolic cylinder $2c = r(1-\cos\theta)$.

In order to find the path described by the vortex, we must subtract $-\tfrac{1}{2}m \log \{(x-x')^2+(y-y')^2\}$ and then put $x = x', y = y'$; we thus obtain

$$\psi = -\tfrac{1}{2}m \log \frac{\pi^2}{8a^2} - \tfrac{1}{2}m \log \frac{\cosh \pi y/a - \cos \pi x/a}{(1 + \cosh \pi y/a)(1 + \cos \pi x/a)}$$
$$= -\tfrac{1}{2}m \log (\sec^2 \pi x/2a - \operatorname{sech}^2 \pi y/2a)$$

neglecting constant terms. Transforming this expression we obtain

$$\epsilon^{-2\psi/m} = \sec^2 \{\tfrac{1}{2}\pi (r/c)^{\frac{1}{2}} \cos \tfrac{1}{2}\theta\} - \operatorname{sech}^2 \{\tfrac{1}{2}\pi (r/c)^{\frac{1}{2}} \sin \tfrac{1}{2}\theta\},$$

which is the equation of the path of a vortex in a parabolic cylinder.

298. Professor Greenhill[1] has shown that the equation of the path of a vortex in a rectangular prism, the origin being at a corner, is

$$\operatorname{ctn}^2 (Kx/a) + \operatorname{ctn}^2 (K'y/b) = \epsilon^{-2\psi/m} - 1,$$

where $2a$, $2b$ are the sides of the rectangular section; $K/a = K'/b$, and the functions of x are to mod. k, whilst those of y are to mod. k'. He has also solved the same problem when the boundaries are two arcs of concentric circles and two radii inclined at an angle π/n.

Coates[2] has shown how Greenhill's expression for the current function due to a vortex situated in a rectangle may be transformed, so as to give the current function due to a vortex in an elliptic cylinder.

299. We shall now find the current function due to a vortex outside an elliptic cylinder.

The method of images is not applicable to problems in which the boundary is elliptic, and we shall therefore solve the problem by means of conjugate functions.

Let ξ, η be conjugate functions such that $x + \iota y = c \cos (\xi - \iota\eta)$; and let (ξ', η') be the co-ordinates of the vortex Q, then if ξ, η be the coordinates of any point P of the liquid,

$$\begin{aligned} QP^2 &= (x - x')^2 + (y - y')^2 \\ &= \{x + \iota y - (x' + \iota y')\}\{x - \iota y - (x' - \iota y')\} \\ &= c^2 \{\cos (\xi - \iota\eta) - \cos (\xi' - \iota\eta')\}\{\cos (\xi + \iota\eta) - \cos (\xi' + \iota\eta')\} \\ &= c^2 \{\cosh (\eta' + \eta) - \cos (\xi' + \xi)\}\{\cosh (\eta' - \eta) - \cos (\xi' - \xi)\}. \end{aligned}$$

[1] *Quart. Journ.* vol. xv. p. 25. [2] *Ibid.* xvi. p. 81.

Now

$$\log\{\cosh(\eta'+\eta)-\cos(\xi'+\xi)\}$$
$$=\log\tfrac{1}{2}+\eta'+\eta+\log\{1-\epsilon^{-\eta'-\eta+\iota(\xi'+\xi)}\}+\log\{1-\epsilon^{-\eta'-\eta-\iota(\xi'+\xi)}\}$$
$$=\log\tfrac{1}{2}+\eta'+\eta-2\Sigma_1^\infty n^{-1}\epsilon^{-n(\eta'+\eta)}\cos n(\xi'+\xi),$$

therefore

$$\log QP=\log\tfrac{1}{2}c+\eta'-\Sigma_1^\infty n^{-1}\{\epsilon^{-n(\eta'+\eta)}\cos n(\xi'+\xi)$$
$$+\epsilon^{-n(\eta'-\eta)}\cos n(\xi'-\xi)\}\ \ldots\ldots\ldots\ldots\ldots(25),$$

This series is always convergent when $\eta'>\eta$. We may therefore put

$$\psi=-m\log QP+\Psi\ \ldots\ldots\ldots\ldots\ldots\ldots(26),$$

where

$$\Psi=m\Sigma_1^\infty n^{-1}\epsilon^{-n(\eta-\beta)}(A_n\cos n\xi+B_n\sin n\xi)+m(\log\tfrac{1}{2}c+\eta'+\eta-\beta).$$

Now $\psi=0$ at the surface where $\eta=\beta$. Substituting these values of ψ and $\log QP$ in (26), and putting $\eta=\beta$, we find

$$-A_n=2\epsilon^{-n\eta'}\cos n\xi'\cosh n\beta$$
$$-B_n=2\epsilon^{-n\eta'}\sin n\xi'\sinh n\beta;$$

therefore

$$\Psi=-m\Sigma_1^\infty n^{-1}\{\epsilon^{-n(\eta+\eta')}\cos n(\xi'+\xi)+\epsilon^{-n(\eta+\eta'-2\beta)}\cos n(\xi'-\xi)\}$$
$$+m(\log\tfrac{1}{2}c+\eta'+\eta-\beta)$$
$$=\tfrac{1}{2}m\log\{\cosh(\eta+\eta')-\cos(\xi+\xi')\}$$
$$+\tfrac{1}{2}m\log\{\cosh(\eta+\eta'-2\beta)-\cos(\xi'-\xi)\}+m\log c\ \ldots(27),$$

therefore

$$\psi=-\tfrac{1}{2}m\log\frac{\cosh(\eta'-\eta)-\cos(\xi'-\xi)}{\cosh(\eta'+\eta-2\beta)-\cos(\xi'-\xi)}\ldots\ldots(28).$$

To find the curve described by the vortex we must put $\eta=\eta'$, $\xi=\xi'$ in (27), whence

$$\Psi=\tfrac{1}{2}m\log c^2\{\cosh 2\eta-\cos 2\xi\}\cosh 2(\eta-\beta)$$

therefore the equation of the path is

$$(\cosh 2\eta-\cos 2\xi)\cosh 2(\eta-\beta)=\text{const.}$$

For further information respecting the images of vortices, and also for other cases of vortex motion in and about elliptic cylinders, the reader is referred to the authorities cited below[1].

[1] Coates, "Vortex motion in and about elliptic cylinders," *Quart. Jour.* vol. xv. p. 356; vol. xvi. p. 81. Hicks, "On functional images in ellipses," *Quart. Jour.* vol. xvii. p. 327.

On the Method of Inversion.

300. We shall now explain a method by means of which we may derive from any hydrodynamical problem in plane vortex motion of which we know the solution, any number of other problems with their solutions.

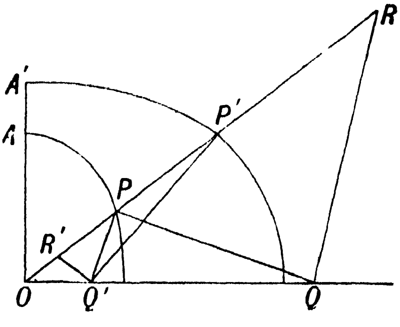
If P be any point on a plane curve S, and O be any fixed point in the plane of S, and if we take another point P' such that $OP \,.\, OP' = a^2$ where a is any constant, the locus of P' is another curve, which is called the inverse of S with respect to O.

301. Let ψ be the current function due to a rectilinear vortex of strength πm situated at any point outside a cylinder whose cross section is S; then if R is the distance of any point from the vortex, we may put

$$\psi = -m \log R + \Psi \quad \text{..................(29)}.$$

At the surface S, ψ is a constant which may be taken to be zero, also Ψ and its first derivatives must be finite and continuous at all points of the liquid, and the derivatives must vanish at infinity. Hence Ψ is the potential of the induced charge when the cylinder is under the action of an electrified line which coincides with the vortex, and which is charged with electricity of line density $\frac{1}{2}m$ per unit of length. Hence the induced charge on the cylinder is equal to $-\frac{1}{2}m$ per unit of length.

302. Let AP be the cross section of the cylinder, $A'P'$ the curve which is the inverse of AP with respect to O; also let R be the electrified wire, which we shall suppose outside AP, ψ the potential of the electric field at Q, and σ the surface density of AP, per unit of length.



Invert the system with respect to O, and let the accented letters refer to the inverse system.

Then

$$\psi = -2\int \sigma \log PQds - m \log RQ \ldots\ldots\ldots\ldots\ldots(30).$$

Also if ψ' be the potential at Q' due to a charge $\frac{1}{2}m$ at R', together with a surface density σ' upon $A'P'$

$$\psi' = -2\int \sigma' \log P'Q'ds' - m \log R'Q'.$$

Now

$$\frac{P'Q'}{PQ} = \frac{OQ'}{OP}, \qquad \frac{R'Q'}{RQ} = \frac{OQ'}{OR}, \text{ and } \frac{ds'}{OP'} = \frac{ds}{OP}.$$

Hence if we take $\sigma' OP' = \sigma OP$, so that $\sigma' ds' = \sigma ds$, we obtain

$$\psi' = -2\int\sigma (\log PQ - \log OP)\, ds - \log OQ' \int \sigma ds$$
$$- m (\log RQ + \log OQ' - \log OR) \ldots\ldots(31).$$

But $$2\int \sigma ds = -m$$

and $$-2\int \sigma \log OPds = -\text{potential of } R \text{ at } O$$
$$= m \log OR$$

by (30). Substituting in (31) we obtain

$$\psi' = -2\int \sigma \log PQds - m \log RQ$$
$$= \psi.$$

Now ψ is zero at all points within AP, therefore ψ' is zero at all points without $A'P'$; hence ψ' is the potential of the electric field, when the inverse cylinder is under the action of an electrified line situated at a point R' within the inverse cylinder, which is the inverse point of R.

If R is inside AP, R' will be outside $A'P'$, and the same results hold good mutatis mutandis.

Hence if we know the current function due to any number of rectilinear vortices which are situated on one side of a cylinder whose cross section is a closed or infinite curve, the method of inversion enables us to obtain the solution for a cylinder, whose cross section is the inverse curve with respect to any point in the plane of the cross section.

303. We can now prove the following proposition.

Let ξ, η be conjugate functions of x, y such that

$$\xi + \iota\eta = f\{(x + \iota y)/c\};$$

and let ξ_1, η_1 be conjugate functions of x_1, y_1, such that $\xi_1 + \iota\eta_1 = f\{a^2/c\,(x_1 - \iota y_1)\}$; also let $F(\xi, \eta)$ be the current function

of a liquid bounded externally or internally by the cylinder $\eta = \beta$, due to a vortex at any point P of the liquid. Then $F(\xi_1, \eta_1)$ will be the current function of a liquid bounded internally or externally by the inverse cylinder $\eta_1 = \beta$, due to a vortex situated at a point P_1 which is the inverse of P with respect to the origin.

If the vortices are replaced by electrified lines, and the cylindrical boundaries by conductors, we have shown that if ψ, ψ_1 be the current functions due to the two hydrodynamical systems, these quantities will be the electric potentials of the two electro-static systems; hence $\psi = \psi_1$.

Let (x, y) be the rectangular and (ξ, η) the curvilinear coordinates of any point Q; and let (x_1, y_1), (ξ_1, η_1) be the coordinates of the inverse point Q_1. Then if a is the constant of inversion,

$$x = a^2 x_1/r_1^2, \quad y = a^2 y_1/r_1^2,$$

therefore $$x + \iota y = a^2/(x_1 - \iota y_1),$$

therefore $$\xi_1 + \iota\eta_1 = f\{a^2/c\,(x_1 - \iota y_1)\} = f\{(x + \iota y)/c\}$$
$$= \xi + \iota\eta,$$

whence $$\xi = \xi_1, \quad \eta = \eta_1.$$

Hence if $$\psi = F(\xi, \eta), \text{ then } \psi_1 = F(\xi_1, \eta_1).$$

304. In § 296 we have found an expression for the current function due to a vortex between two parallel planes, and by means of the preceding proposition we can obtain the current function due to a vortex in a liquid which is bounded by two circular cylinders. Also if in the expression in § 297 for the path of a vortex within a parabolic cylinder we write c/r for r/c, the resulting expression will give the path of a vortex in a liquid bounded internally by a cylinder whose cross section is a cardioid.

The expression for the current function due to a vortex outside an elliptic cylinder, is the expression for a vortex within a cylinder whose cross section is the inverse of an ellipse with respect to its centre or focus; but in the former case $\xi + \iota\eta = \sec^{-1}(x + \iota y)/c$, and in the latter it equals $2\sec^{-1}\{(x + \iota y)/2c\}^{\frac{1}{2}}$.

The expression found by Coates for the current function due to a vortex inside an elliptic cylinder, similarly determines the current function due to a vortex in a liquid bounded internally by a cylinder whose cross section is the inverse of an ellipse with respect to its centre or focus.

EXAMPLES.

1. If the axis of a hollow vortex be the axis of z, measured vertically downwards, the plane of xy being the asymptotic plane to the free surface, and if ϖ be the atmospheric pressure: prove that the equation of the surface at which the pressure is $\varpi + g\rho a$ is

$$(x^2 + y^2)(z - a) = c^3,$$

where c is a constant.

2. Three rectilinear vortices of equal strengths form the edges of an equilateral triangular prism. Prove that they will always form the three edges of an equal prism.

3. The space between two infinite parallel planes distant c from each other is filled with water. Half way between the planes is placed a rectilinear vortex. Prove that the path of any particle of water is given by the equation

$$\cosh \pi y/c = A \cos \pi x/c,$$

the axis of x being perpendicular to the planes.

Prove also that the velocity potential is

$$m \tan^{-1} (\sinh \pi y/c \operatorname{cosec} \pi x/c).$$

4. An infinite plane vortex sheet in which the rotation is everywhere the same in magnitude and direction exists in an infinite mass of liquid; prove that the resultant velocity at any point (x, y, z) is

$$-\frac{q}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x\,dy'\,dz'}{\{x^2 + (y - y')^2 + (z - z')^2\}^{\frac{3}{2}}},$$

where yz is the plane of the vortex sheet, the axis of z is parallel to the axis of molecular rotation, and q is the product of the section by the angular velocity for each line.

Evaluate this integral, and explain the result.

5. If n rectilinear vortex filaments of equal strengths, be initially at the angles of a prism whose base is a regular polygon of n sides, show that they will always be so situated, and that each filament will describe the circumscribed cylinder with velocity $k(n-1)/2a$ where k is the velocity due to each vortex at unit distance and a is the radius of the cylinder. Show also that the equation of the relative stream lines referred to the radius through a vortex as initial line is $r^{2n} - 2a^n r^n \cos n\theta - b^{2n} = 0$.

6. The space on one side of the concave branch of a rectangular hyperbolic cylinder is filled with liquid, and a rectilinear vortex exists in the liquid; prove that it moves in a cylinder having the same asymptotic planes as the boundary.

7. The motion of a liquid in two dimensions is such that the vorticity ζ is constant; prove that the general functional equation of the stream lines is

$$\phi(y+\iota x)+\chi(y-\iota x)-\tfrac{1}{2}\zeta(x^2+y^2)=c.$$

Prove that if the space between one branch of the hyperbola $x^2-3y^2=a^2$ and the tangent to its vertex be filled with liquid, it will be possible for the liquid to move steadily with constant vorticity, and find the form of the stream lines.

8. A mass of liquid whose outer boundary is an infinitely long cylinder of radius b, is in a state of cyclic irrotational motion and is under the action of a uniform pressure Π over its external surface. Prove that there must be a concentric cylindrical hollow whose radius a is determined by the equation

$$8\pi^3a^2b^2\Pi = M\kappa^2,$$

where M is the mass of unit length of the liquid, and κ is the circulation.

If the cylinder receive a small symmetrical displacement, prove that the time of a small oscillation is

$$\frac{4}{\kappa}\pi^2a^2b^2\sqrt{\frac{\log b/a}{b^4-a^4}}.$$

9. A fixed cylinder of radius a is surrounded by incompressible homogeneous fluid extending to infinity. Symmetrically arranged round it as generators on a cylinder of radius c ($> a$) coaxial with the given one, are n rectilinear vortex filaments each of strength m. Show that the filaments will remain on this cylinder throughout the motion, and will revolve round its axis with angular velocity

$$\frac{m}{2\pi c^2}\cdot\frac{(n+1)c^{2n}+(n-1)a^{2n}}{c^{2n}-a^{2n}},$$

and that the velocity of any point P of the fluid is

$$\frac{mnr^{n-1}}{\pi}\cdot\frac{c^n-b^n}{(r^{2n}-2c^nr^n\cos n\theta+c^{2n})(r^{2n}-2b^nr^n\cos n\theta+b^{2n})},$$

where $a^2=bc$, r is the distance of P from the axis, and θ is the angle between a plane containing P and the axis, and a plane containing P and the instantaneous position of any one of the filaments.

10. Four straight vortex filaments with alternately positive and negative rotations are placed symmetrically within a cylinder filled with liquid; prove that if the motion is steady the distance of each filament from the axis of the cylinder is nearly three-fifths of the radius of the latter.

11. Prove that three infinitely long straight cylindrical vortices of equal strengths will be in stable steady motion, when situated at the vertices of an equilateral triangle whose sides are large compared with the radii of the sections of the vortices; and that if they are slightly displaced, prove that the time of a small oscillation is the same as that of the time of revolution of the system in its undisturbed state.

12. A straight cylindrical vortex column of uniform vorticity ζ, is surrounded by an infinite quantity of liquid moving irrotationally which is at rest at infinity; prove that the difference between the kinetic energy included between two planes at right angles to the axis of the cylinder and separated by unit distance, when the cross section is an ellipse, and when it is a circle of equal area A is

$$\rho\pi^{-1}\zeta^2 A^2 \log (a+b)/2\sqrt{ab},$$

where ρ is the density of the liquid, and a and b are the semiaxes of the ellipse.

13. Examine the stability of Kirchhoff's elliptic vortex, when the cross section of the vortex column is displaced into a curve slightly different from an ellipse.

14. Prove or verify that the current function due to a stationary vortex situated at the centre of an elliptic cylinder, is

$$\psi = -\tfrac{1}{2}m \log 4c^2 \operatorname{sn} u \operatorname{sn}(u-K) \operatorname{sn} v \operatorname{sn}(v-K),$$

where $$\xi + \iota\eta = u,\ \xi - \iota\eta = v.$$

Prove also that the velocity potential is

$$\phi = m \tan^{-1} \frac{k' \operatorname{sn}(2K\xi/\pi) \operatorname{sn}(K'\eta/\beta)}{\operatorname{cn}(2K\xi/\pi)},$$

where $\beta = \tfrac{1}{2}\pi K'/K$ is the value of η at the cylindrical boundary; and the functions of ξ are to mod. k, and those of η to mod. k'.

15. A quantity of liquid whose vorticity is uniform and equal to ζ, and whose external surface is a circular cylinder, surrounds a concentric cylinder of radius a. The external surface is subjected to a constant pressure Π. Prove that if the inner cylinder be removed, the velocity of the internal surface when its radius is α, is equal to

$$\frac{1}{\alpha}\sqrt{\frac{(a^2 - \alpha^2)(\zeta^2 c^2 - 2\Pi/\rho)}{\log a^2/(\alpha^2 + c^2)}},$$

where $\pi\rho c^2$ is the mass of the liquid per unit of length.

16. If a vortex is moving in a liquid bounded by a fixed plane, prove that a stream line can never coincide with a line of constant pressure.

17. If a pair of equal and opposite vortices are situated inside or outside a circular cylinder of radius a, prove that the equation of the curve described by each vortex is,

$$(r^2 - a^2)^2 (r^2 \sin^2 \theta - b^2) = 4a^2 b^2 r^2 \sin^2 \theta,$$

where b is a constant.

CHAPTER XIV.

CIRCULAR VORTICES.

305. A CIRCULAR vortex ring may be supposed to be made up of a large number of indefinitely thin circular vortex filaments, every element of which is rotating with angular velocity ω about the tangent to the circle of which the element forms a part.

We have shown in Chapter IV. that the velocity due to a fine vortex filament, is proportional to the magnetic force exerted by an electric current, which flows along a fine wire which coincides with the vortex; and it has been shown by Maxwell[1], that if electric currents flow round an anchor ring of small cross section, the effect is the same as if the currents were condensed into a single one flowing along the central line of the ring. If therefore the cross section of the ring is small in comparison with its aperture, the effect of the ring upon the irrotationally moving liquid by which it is surrounded, will be approximately the same as that of a fine vortex filament of equal strength, which coincides with the central line of the core. Hence rings of small cross section may be approximately regarded as vortex filaments, and we may disregard the effects which are due to any deformation of the form of the cross section, or to anything which takes place within the substance of the ring. We shall thereby greatly simplify the analysis; but when we wish to ascertain what goes on inside the ring, it will be necessary to employ toroidal functions, and the investigation becomes much more complicated.

[1] *Electricity and Magnetism*, 2nd edition, vol. II. § 683.

306. Let us in the first place confine our attention to a single circular vortex of small cross section in an infinite liquid. It is clear that the motion is symmetrical with respect to a line passing through the centre of the ring and perpendicular to its plane, which we shall choose as the axis of z. Hence by § 38 (28), if ω be the molecular rotation, Stokes' current function satisfies the equation

$$\frac{d^2\psi'}{dz^2}+\frac{d^2\psi'}{d\varpi^2}-\frac{1}{\varpi}\frac{d\psi'}{d\varpi}+2\varpi\omega=0,$$

at all points in the interior of the ring. Outside the ring the current function satisfies the equation

$$\frac{d^2\psi}{dz^2}+\frac{d^2\psi}{d\varpi^2}-\frac{1}{\varpi}\frac{d\psi}{d\varpi}=0.$$

Putting $\psi=\chi\varpi$ these equations become

$$\frac{d^2\chi'}{dz^2}+\frac{d^2\chi'}{d\varpi^2}+\frac{1}{\varpi}\frac{d\chi'}{d\varpi}-\frac{\chi'}{\varpi^2}+2\omega=0 \quad \text{...........(1)}$$

inside; and

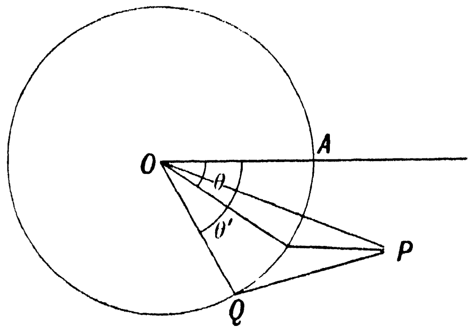
$$\frac{d^2\chi}{dz^2}+\frac{d^2\chi}{d\varpi^2}+\frac{1}{\varpi}\frac{d\chi}{d\varpi}-\frac{\chi}{\varpi^2}=0 \quad \text{.................(2)}$$

outside.

These equations show that $\chi\cos\theta$ is the potential of a distribution of matter of density $\omega\cos\theta./2\pi$, which occupies the same portion of space as the vortex ring.

On account of the smallness of the cross section, ω may be treated as approximately constant, and $\chi\cos\theta$ will be the potential of a fine circular wire whose density is $\omega\cos\theta./2\pi$, θ being measured from some fixed point on the ring.

307. To find this potential, let O be the centre of the vortex ring, and let the axis of z be perpendicular to the plane of the



paper; let A be the fixed point on the ring from which θ is measured, and let P be any point whose coordinates are z, ϖ, θ;

also let Q be any point on the central line of the vortex whose coordinates are z', a, θ'; then if σ be the cross section

$$\chi \cos\theta = \frac{a\omega\sigma}{2\pi}\int_{\theta}^{2\pi+\theta} \frac{\cos\theta'\,d\theta'}{\{(z-z')^2+\varpi^2+a^2-2\varpi a\cos(\theta'-\theta)\}^{\frac{1}{2}}}.$$

Putting $\theta'-\theta=\epsilon$, we obtain

$$\chi\cos\theta = \frac{\sigma\omega a}{2\pi}\int_0^{2\pi}\frac{(\cos\theta\cos\epsilon-\sin\theta\sin\epsilon)\,d\epsilon}{\{(z-z')^2+\varpi^2+a^2-2\varpi a\cos\epsilon\}^{\frac{1}{2}}}.$$

The second integral vanishes, whence

$$\psi = \chi\varpi = \frac{\sigma\omega\varpi a}{\pi}\int_0^{\pi}\frac{\cos\epsilon\,d\epsilon}{\{(z-z')^2+\varpi^2+a^2-2\varpi a\cos\epsilon\}^{\frac{1}{2}}}\ \ldots(3),$$

which determines the value of ψ at any point outside the vortex.

308. We can now determine the motion of the vortex. Putting

$$k'^2 = \frac{4\varpi a}{(z-z')^2+(\varpi+a)^2},\quad 2\eta=\epsilon,$$

(3) becomes

$$\psi = \sigma\omega\pi^{-1}k'(\varpi a)^{\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\frac{2\cos^2\eta-1}{(1-k'^2\cos^2\eta)^{\frac{1}{2}}}\,d\eta$$

$$= \sigma\omega\pi^{-1}(\varpi a)^{\frac{1}{2}}\{2(F'-E')/k'-k'F'\}\ \ \ldots\ldots\ldots(4).$$

Putting

$$U = 2(F'-E')/k' - k'F,\ m=\sigma\omega,$$

where m is the strength of the vortex, we obtain

$$\psi = m(\varpi a)^{\frac{1}{2}}U/\pi.$$

At the surface of the vortex ring, z and ϖ are very nearly equal to z' and a respectively, hence k' is very nearly equal to unity; whence if $L=\log 4/k$, we have approximately[1]

$$F' = L + \tfrac{1}{4}k^2(L-1),$$

$$E' = 1 + \tfrac{1}{2}k^2(L-\tfrac{1}{2}),$$

therefore

$$U = L - 2 + \tfrac{3}{4}k^2(L-1).$$

Also

$$w = \frac{1}{\varpi}\frac{d\psi}{d\varpi} = \frac{m}{\pi}\sqrt{\frac{a}{\varpi}}\left(\frac{dU}{d\varpi}+\frac{U}{2\varpi}\right),$$

$$u = -\frac{1}{\varpi}\frac{d\psi}{dz} = -\frac{m}{\pi}\sqrt{\frac{a}{\varpi}}\frac{dU}{dz}.$$

[1] Cayley, *Elliptic Functions*, p. 54.

Now $$k^2 = \frac{(z-z')^2 + (\varpi - a)^2}{(z-z')^2 + (\varpi + a)^2},$$

whence if e be the radius of the cross section of the ring, we have approximately at the surface of the ring

$$k = \frac{e}{2a}, \quad \frac{dk}{d\varpi} = \frac{2a(\varpi - a) - e^2}{4a^2 e}, \quad \frac{dk}{dz} = \frac{z-z'}{2ae}.$$

And $$\frac{dU}{dk} = -\frac{1}{k} + \tfrac{3}{2}k(L - \tfrac{3}{2})$$

$$= -\frac{2a}{e} + \frac{3e}{4a}(L - \tfrac{3}{2}).$$

Therefore $$u = \frac{m}{\pi}\left\{\frac{2a}{e} - \frac{3e}{4a}(L - \tfrac{3}{2})\right\}\frac{z-z'}{2ae}.$$

When $z = z'$, $u = 0$, hence the radius of the ring remains invariable.

Again $$w = -\frac{m}{\pi}\left\{\frac{2a}{e} - \frac{3e}{4a}(L - \tfrac{3}{2})\right\}\frac{2a(\varpi - a) - e^2}{4a^2 e} + \frac{m}{2\pi a}(L - 2).$$

In order to obtain the velocity of translation of the ring we must put $\varpi = a$, and we obtain

$$w = \frac{m}{2\pi a}(L - 1)$$

$$= \frac{m}{2\pi a}\left(\log\frac{8a}{e} - 1\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

which shows that the ring moves forward in the direction of the cyclic motion through its aperture with constant velocity.

By § 61 every element of the vortex produces a velocity at the centre of the ring which is equal to $mds/2\pi a^2$; hence the velocity at the centre is equal to $m/a = \pi e^2\omega/a$.

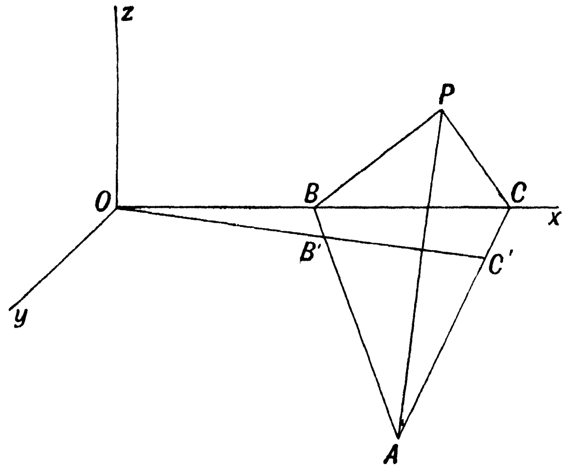
Hence an isolated circular vortex in an infinite liquid moves without sensible change of size in a direction which is perpendicular to its plane, with a constant velocity, which is small compared with that of the liquid in the immediate neighbourhood of its central line, but large compared with the velocity of the liquid at the centre of the ring.

309. Let us now consider the motion of two parallel circular vortices whose centres lie on the axis of z. If the directions of molecular rotation are the same in both, the effect of the hindermost vortex on the one in advance, will be to increase the radius

and retard the velocity of the latter; whilst the effect of the foremost vortex upon the one in the rear, will be to diminish the radius and increase the velocity of the latter. Hence the hindermost ring will overtake and shoot through the foremost; after which the circumstances will be reversed, and the one which is now in the rear will overtake and shoot through the one in advance.

310. If the directions of rotation are in opposite directions the rings will either recede from, or advance towards one another. If the former is the case the radii of each ring will diminish, whilst the converse will be the case if the rings are advancing towards one another. In the latter case the velocity of approach continually diminishes whilst the radii of the rings increase; also if the vortices are of equal size and strength, there will be no flux across a fixed plane parallel to them and bisecting the distance between them, and we may therefore remove one of the vortices and substitute for it a rigid plane boundary. Hence the motion of a vortex which is moving in a liquid towards or from a fixed rigid plane, is obtained by substituting for the plane a second vortex of equal size and opposite strength, which is the image of the first with respect to the plane.

311. We shall now determine the image of a circular vortex in a sphere[1].



We shall in the first place show that every element ds of a vortex ring within the sphere, together with a corresponding

[1] Lewis, *Quart. Journ.* vol. XVI. p. 338.

element ds' without the sphere, which occupies the position of the electrical image of ds, will produce over the surface of the sphere a velocity which is everywhere tangential, provided certain other conditions are satisfied.

Let O be the centre of the sphere, and let BB', CC' be the two elements, m, m' their strengths, and let the plane in which they lie be the plane of xy. Since $OB \, . \, OC = OB' \, . \, OC'$, the angle $ACB = AB'C' = ABC$ ultimately, whence $AC = AB$.

Let OC be the axis of x, $BP = r$, $CP = r'$, $OB = f$, $OC = f'$, $ABC = \theta$, also let β and γ be the angles which the planes APB and APC respectively make with the plane xy.

Let (x, y, z) be the coordinates of P, and u, v, w the velocities at P; then by § 61 if the two vortex elements at B and C are parts of complete filaments

$$2\pi u = mr^{-2}\, ds \sin B \sin \beta \sin \theta + m'r'^{-2}\, ds' \sin C \sin \gamma \sin \theta,$$

$$2\pi v = -\, mr^{-2}\, ds \sin B \sin \beta \cos \theta + m'r'^{-2}\, ds' \sin C \sin \gamma \cos \theta,$$

$$2\pi w = -\, mr^{-2}\, ds \sin B \cos \beta + m'r'^{-2}\, ds' \sin C \cos \gamma.$$

But

$$z = r \sin B \sin \beta = r' \sin C \sin \gamma,$$

$$(x - f) \sin \theta - y \cos \theta = z \cot \beta,$$

$$(f' - x) \sin \theta - y \cos \theta = z \cot \gamma.$$

Therefore

$$2\pi u = (mr^{-3}\, ds + m'r'^{-3}\, ds')\, z \sin \theta,$$

$$2\pi v = (-\, mr^{-3}\, ds + m'r'^{-3}\, ds')\, z \cos \theta,$$

$$2\pi w = (-\, mr^{-3}\, ds \cot \beta + m'r'^{-3}\, ds' \cot \gamma)\, z.$$

In order that there may be no flux across the sphere, we must have at the surface

$$ux + vy + wz = 0.$$

Therefore

$$mr^{-3}\, ds\, (x \sin \theta - y \cos \theta - z \cot \beta)$$
$$+\, m'r'^{-3}\, ds'\, (x \sin \theta + y \cos \theta + z \cot \gamma) = 0,$$

whence $$mfr^{-3}\, ds = -\, m'f'r'^{-3}\, ds'.$$

But $ds/f = ds'/f'$; and $(r/r')^2 = f/f' = (f/a)^2$, where a is the radius of the sphere; therefore

$$m\sqrt{f} = -\, m'\sqrt{f'}.$$

Hence the molecular rotations of the two vortex elements must be in opposite directions, and their strengths must vary inversely as the square roots of the distances of the two elements from the centre of the sphere. Now along each ring m is constant, also since $f/f' = (f/a)^2, f$ must be constant, and therefore each vortex ring must lie on a sphere concentric with the sphere which forms the boundary of the liquid.

312. We have shown in § 62 that the velocity potential at any point due to a fine vortex is equal to $-m\Omega/2\pi$, where Ω is the solid angle subtended by the vortex at the point. When the vortex is circular, this solid angle may be easily expressed in a series of spherical harmonics[1], and we may thus obtain the expressions for the component velocities in the form of a series. This method of proceeding is especially useful, when we desire to obtain the effect of a vortex at a point very distant from it, for in this case a few terms of the series will be sufficient.

We could also apply this method to find the velocity potential due to a vortex situated outside a fixed sphere, but the preceding investigation shows that the series representing the image will not be the velocity potential of a single vortex unless the original vortex lies on a concentric sphere; when this is not the case, the image will consist of a hydrodynamical system of more or less complexity, which will be dependent on the form and position of the original vortex ring.

In considering the motion of two vortices we have supposed that their planes are parallel, and that their centres lie on a straight line which is perpendicular to their planes. For the discussion of the motion of two vortex rings whose planes are not parallel, we must refer the reader to Part II. of Prof. J. J. Thomson's *Motion of Vortex Rings*.

[1] Ferrers, *Spherical Harmonics*, ch. III. Maxwell, *Electricity and Magnetism*, vol. II. ch. XIV.

Vibrations of a Circular Vortex Ring[1].

313. The vibrations to which a vortex ring may be subject, may be divided into two classes, vibrations which involve a deformation of the surface of the ring, without any deformation of the central line; and vibrations which involve a deformation of the central line as well as a deformation of the surface of the ring.

A complete investigation of the stability of a vortex which is in a state of steady motion or kinetic equilibrium, would involve the consideration of the problem in its most general form. When however the cross section of the ring is small in comparison with its aperture, we may without sensible error treat these two kinds of vibrations separately. We shall therefore in the present section confine our attention to vibrations involving a deformation of the central line alone, and shall neglect deformations of the surface. In the closing portion of this Chapter, we shall suppose that the central line retains its circular form, and investigate what may be called *fluted vibrations*, that is to say vibrations which consist of trains of waves travelling over the surface of the ring, whose crests are circles parallel to the central line.

314. Let a be the radius of the central line when the ring is undisturbed, $\mathfrak{z}$ its distance from the origin; and let x', y', z' be rectangular, and ϖ', ψ, z' cylindrical coordinates of any point on the central line during the disturbed motion; also let x, y, $\mathfrak{z}+\zeta$ be rectangular, and ϖ, θ, $\mathfrak{z}+\zeta$ be cylindrical coordinates of any point of space. Let

$$\varpi' = a + \alpha_n \cos n\psi, \quad z' = \mathfrak{z} + \gamma_n \cos n\psi \quad \ldots\ldots\ldots\ldots (6),$$

where in the beginning of the disturbed motion, α_n, γ_n are small functions of the time, whose squares and products may be neglected. Then

$$x' = \varpi' \cos\psi, \quad y' = \varpi' \sin\psi,$$

whence

$$\left.\begin{aligned} dx'/d\psi &= -a\sin\psi - \alpha_n(\cos n\psi \sin\psi + n\sin n\psi\cos\psi) \\ dy'/d\psi &= a\cos\psi + \alpha_n(\cos n\psi\cos\psi - n\sin n\psi\sin\psi) \\ dz'/d\psi &= -n\gamma_n \sin n\psi \end{aligned}\right\} \ldots (7).$$

[1] J. J. Thomson, *Phil. Trans.* 1882, and *Motion of Vortex Rings*, Part I.

Let r be the distance between the points $(x, y, \mathfrak{z}+\zeta)$ and (x', y', z'), also let $\zeta' = \zeta - \gamma_n \cos n\psi$, then

$$r^2 = \varpi^2 + \varpi'^2 + \zeta'^2 - 2\varpi\varpi' \cos(\psi - \theta).$$

Now r^{-3} can evidently be expanded in a series of cosines of multiples of $\psi - \theta$, we may therefore put

$$r^{-3} = C_0 + C_1 \cos(\psi - \theta) + \ldots\ldots C_m \cos m(\psi - \theta),$$

where the C's are functions of ϖ, ϖ', ζ'. Since ψ enters into ϖ', ζ' in the forms $\alpha_n \cos n\psi$, $\gamma_n \cos n\psi$, the terms in the C's which involve ψ will be small quantities, whence if

$$\{\varpi^2 + a^2 + \zeta^2 - 2\varpi a \cos(\psi - \theta)\}^{-\frac{3}{2}} = A_0 + A_1 \cos(\psi - \theta) + \ldots\ldots A_m \cos m(\psi - \theta) + \ldots \quad (8),$$

we shall have

$$C_m = A_m + \frac{dA_m}{da}\alpha_n \cos n\psi - 2\zeta \frac{dA_m}{d(\zeta^2)}\gamma_n \cos n\psi.$$

In the present investigation ζ will be a small quantity, and we may therefore neglect the last term, we thus obtain

$$C_m = A_m + \frac{dA_m}{da}\alpha_n \cos n\psi \ldots\ldots\ldots\ldots\ldots\ldots \quad (9).$$

315. We must now calculate the velocity due to the vortex during the disturbed motion.

By § 61 the velocity parallel to x of a vortex of strength m is

$$u = \frac{m}{2\pi}\int_0^{2\pi} \frac{1}{r^3}\left\{(z - z')\frac{dy'}{d\psi} - (y - y')\frac{dz'}{d\psi}\right\} d\psi.$$

Substituting the values of $z - z'$ &c. in terms of ψ and neglecting squares of small quantities, the term in brackets becomes

$$\zeta a \cos\psi + ny\gamma_n \sin n\psi + \tfrac{1}{2}a\gamma_n \{(n-1)\cos(n+1)\psi - (n+1)\cos(n-1)\psi\}.$$

Since every term of this expression is small, we may write A for C in the expression for r^{-3}, whence remembering that

$$\int_0^{2\pi} \cos m\psi \cos n\psi d\psi = 0 \text{ or } \pi$$

according as m is unequal or equal to n, we obtain

$$u = \tfrac{1}{2}m\,[\zeta a A_1 \cos\theta + \tfrac{1}{2}a\gamma_n \{(n-1)A_{n+1}\cos(n+1)\theta - (n+1)A_{n-1}\cos(n-1)\theta\} + \tfrac{1}{2}n\varpi\gamma_n A_n \{\cos(n-1)\theta - \cos(n+1)\theta\}] \ldots\ldots\ldots\ldots \quad (10).$$

The velocity parallel to y is

$$v=\frac{m}{2\pi}\int_0^{2\pi}\frac{1}{r^3}\left\{(x-x')\frac{dz'}{d\psi}-(z-z')\frac{dx'}{d\psi}\right\}d\psi.$$

The term in brackets

$$=\zeta a\sin\psi-nx\gamma_n\sin n\psi$$
$$+\tfrac{1}{2}a\gamma_n\{(n-1)\sin(n+1)\psi+(n+1)\sin(n-1)\psi\},$$

whence

$$v=\tfrac{1}{2}m\,[\zeta aA_1\sin\theta$$
$$+\tfrac{1}{2}a\gamma_n\{(n-1)A_{n+1}\sin(n+1)\theta+(n+1)A_{n-1}\sin(n-1)\theta\}$$
$$-\tfrac{1}{2}n\varpi\gamma_nA_n\{\sin(n+1)\theta+\sin(n-1)\theta\}]\text{..............}\quad(11).$$

The velocity parallel to z is

$$w=\frac{m}{2\pi}\int_0^{2\pi}\frac{1}{r^3}\left\{(y-y')\frac{dx'}{d\psi}-(x-x')\frac{dy'}{d\psi}\right\}d\psi.$$

The expression in brackets is equal to

$$a^2-a(x\cos\psi+y\sin\psi)$$
$$+2a\alpha_n\cos n\psi-\tfrac{1}{2}y\alpha_n\{(n+1)\sin(n+1)\psi+(n-1)\sin(n-1)\psi\}$$
$$-\tfrac{1}{2}x\alpha_n\{(n+1)\cos(n+1)\psi-(n-1)\cos(n-1)\psi\}.$$

Since the first two terms are not multiplied by any small quantity, we must not put A for C in the value of r^{-3} by which these terms are multiplied, but must employ the value of C given by (9); whence on integration

$$\text{the 1st term}=\tfrac{1}{2}ma^2\left(2A_0+\frac{dA_n}{da}\alpha_n\cos n\theta\right),$$

the 2nd term

$$=-\tfrac{1}{2}mA_1a\varpi-\tfrac{1}{4}max\alpha_n\left\{\frac{dA_{n+1}}{da}\cos(n+1)\theta+\frac{dA_{n-1}}{da}\cos(n-1)\theta\right\}$$
$$-\tfrac{1}{4}may\alpha_n\left\{\frac{dA_{n+1}}{da}\sin(n+1)\theta-\frac{dA_{n-1}}{da}\sin(n-1)\theta\right\},$$

and the other terms

$$=\tfrac{1}{2}m\,[2a\alpha_nA_n\cos n\theta$$
$$-\tfrac{1}{2}y\alpha_n\{(n+1)A_{n+1}\sin(n+1)\theta+(n-1)A_{n-1}\sin(n-1)\theta\}$$
$$-\tfrac{1}{2}x\alpha_n\{(n+1)A_{n+1}\cos(n+1)\theta-(n-1)A_{n-1}\cos(n-1)\theta\}].$$

Collecting our results we obtain

$$w=\tfrac{1}{2}m\Big[2A_0a^2-a\varpi A_1$$
$$+\{2A_na+\tfrac{1}{2}\varpi[(n-1)A_{n-1}-(n+1)A_{n+1}]\}\alpha_n\cos n\theta$$
$$+\left\{a\frac{dA_n}{da}-\tfrac{1}{2}\varpi\frac{d}{da}(A_{n+1}+A_{n-1})\right\}a\alpha_n\cos n\theta\Big]\text{...........}\quad(12).$$

316. Having obtained the values of the velocities we can now find the values of $\dot{\alpha}_n$, $\dot{\gamma}_n$.

If e be the radius of the cross section (which is supposed to be very small) the equations of the surface of the ring are

$$\varpi = a + \alpha_n \cos n\theta + e \cos \phi \qquad (13),$$

$$z = \zeta + \gamma_n \cos n\theta + e \sin \phi \qquad (14).$$

By § 12, if $F(\varpi, \theta, \phi, t)$ be the equation of a surface which always contains the same elements of fluid,

$$\frac{dF}{dt} + R\frac{dF}{d\varpi} + \Theta\frac{dF}{d\theta} + \Phi\frac{dF}{d\phi} = 0,$$

where R is the velocity along ϖ, and Θ, Φ are the angular velocities in the directions in which these quantities increase.

Applying this to (13) we obtain

$$\dot{\alpha}_n \cos n\theta - R - n\alpha_n \Theta \sin n\theta - e\Phi \sin \phi = 0.$$

If the motion were undisturbed Θ would be zero, hence in the beginning of the disturbed motion Θ must be a small quantity; the third term is consequently of the second order and may therefore be neglected. We thus obtain

$$R = \dot{\alpha}_n \cos n\theta - e\Phi \sin \phi \qquad (15).$$

But

$$R = u \cos \theta + v \sin \theta$$
$$= \tfrac{1}{2}m\,[\zeta a A_1 + \tfrac{1}{2}a\gamma_n \{(n-1)A_{n+1} - (n+1)A_{n-1}\} \cos n\theta] \ldots (16),$$

by (10) and (11). In this expression $\zeta = \gamma_n \cos n\theta + e \sin \phi$; also the values of the A's must be obtained from (8) by putting

$$\varpi = a + \alpha_n \cos n\theta + e \cos \phi$$

and giving to ζ the above value. Let S_n denote the value of A_n at the surface of the undisturbed vortex, that is when $\alpha_n = \gamma_n = 0$. Then by proceeding in the same manner as in the case of equation (9), we see that

$$A_n = S_n + \frac{dS_n}{d\varpi}\alpha_n \cos n\theta \qquad (17).$$

But since each of the A's is multiplied by a small quantity in (16), we may put S for A, and we thus obtain

$$R = \tfrac{1}{2}m\,[aS_1(\gamma_n \cos n\theta + e \sin \phi)$$
$$+ \tfrac{1}{2}a\gamma_n \{(n-1)S_{n+1} - (n+1)S_{n-1}\} \cos n\theta] \qquad (18).$$

Equating coefficients of $\cos n\theta$ and $\sin\phi$ in the two values of R given by (15) and (18) we obtain

$$\dot{\alpha}_n = \tfrac{1}{2} ma\gamma_n \left[S_1 + \tfrac{1}{2}\{(n-1)S_{n+1} - (n-1)S_{n-1}\}\right] \quad \text{......(19).}$$

$$\Phi = -\tfrac{1}{2} ma\, S_1 \quad \text{...... (20).}$$

Again, the condition that (14) should be always composed of the same elements of fluid, is

$$\frac{dF}{dt} + (w - \dot{\zeta})\frac{dF}{dz} + \Theta\frac{dF}{d\theta} + \Phi\frac{dF}{d\phi} = 0,$$

whence

$$w = \dot{\zeta} + \dot{\gamma}_n \cos n\theta + e\Phi\cos\phi \quad \text{...... (21).}$$

Equating the right-hand side of this equation to the value of w given by (12), we obtain

$$\tfrac{1}{2} m \Big[2A_0 a^2 - a\varpi A_1 + \{2A_n a + \tfrac{1}{2}\varpi[(n-1)A_{n+1} - (n+1)A_{n-1}]\}\alpha_n \cos n\theta$$
$$+ \left\{a\frac{dA_n}{da} - \tfrac{1}{2}\varpi\frac{d}{da}(A_{n+1} + A_{n-1})\right\} a\alpha_n \cos n\theta\Big] = \dot{\zeta} + \dot{\gamma}_n \cos n\theta$$
$$+ e\Phi\cos\phi \quad \text{...(22).}$$

Since the last two terms on the left-hand side are multiplied by α_n, we may put $\varpi = a$, $A_n = S_n$; but in the first two terms which are not multiplied by a small quantity, we must substitute for A_n its value from (17), and for ϖ its value from (13). Making these substitutions and equating coefficients we obtain

$$\dot{\zeta} = \tfrac{1}{2} ma^2 (2S_0 - S_1) \quad \text{...... (23),}$$

$$\Phi = -\tfrac{1}{2} maS_1 \quad \text{...... (24).}$$

$$\dot{\gamma}_n = \tfrac{1}{2} ma\alpha_n \Big[2S_n - S_1 + \tfrac{1}{2}\{(n-1)S_{n-1} - (n+1)S_{n+1}\}$$
$$+ a\frac{d}{da}\left\{S_n - \tfrac{1}{2}(S_{n+1} + S_{n-1})\right\} + a\frac{d}{d\varpi}(2S_0 - S_1)\Big] \quad \text{...(25).}$$

317. We must now calculate the S's. From (8) it follows that if we put

$$\varpi = a + e\cos\phi,\ \zeta = e\sin\phi,\ 2\varpi aq = \varpi^2 + a^2 + \zeta^2,$$

then

$$S_n = \frac{1}{\pi(2\varpi a)^{\frac{3}{2}}}\int_0^{2\pi} \frac{\cos n\theta\, d\theta}{(q - \cos\theta)^{\frac{3}{2}}},\quad 2S_0 = \frac{1}{\pi(2\varpi a)^{\frac{3}{2}}}\int_0^{2\pi} \frac{d\theta}{(q - \cos\theta)^{\frac{3}{2}}}.$$

Since e is very small, q is nearly equal to unity, and we there-

fore require the values of the preceding integrals, when q is nearly equal to unity. Let

$$c_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos n\theta d\theta}{(q-\cos\theta)^{\frac{3}{2}}}, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos n\theta d\theta}{(q-\cos\theta)^{\frac{1}{2}}}.$$

From § 277, it appears that b_n is a zonal toroidal function of the second kind; also since $c_n = -2db_n/dq$, it follows from § 273 (55) and § 275 that

$$c_n = \frac{2n+1}{q^2-1}(qb_n - b_{n+1}) \dots\dots\dots\dots\dots\dots (26),$$

and that

$$(1-q^2)\frac{d^2b_n}{dq^2} - 2q\frac{db_n}{dq} + (n^2-\tfrac{1}{4})\,b_n = 0 \dots\dots\dots (27).$$

In order to find the value of b_n when q is nearly equal to unity, assume

$$b_n = \phi(q)\log\frac{q-1}{16(q+1)} + \psi(q).$$

Substituting in (27) we obtain

$$(1-q^2)\frac{d^2\phi}{dq^2} - 2q\frac{d\phi}{dq} + (n^2-\tfrac{1}{4})\,\phi = 0,$$

$$-4\frac{d\phi}{dq} + (1-q^2)\frac{d^2\psi}{dq^2} - 2q\frac{d\psi}{dq} + (n^2-\tfrac{1}{4})\,\psi = 0.$$

In these equations put $x = q-1$, and they become

$$x(2+x)\frac{d^2\phi}{dx^2} + 2(1+x)\frac{d\phi}{dx} - (n^2-\tfrac{1}{4})\,\phi = 0 \dots\dots (28),$$

$$4\frac{d\phi}{dx} + x(2+x)\frac{d^2\psi}{dx^2} + 2(1+x)\frac{d\psi}{dx} - (n^2-\tfrac{1}{4})\,\psi = 0 \dots (29).$$

In order to solve (28), assume $\phi = \Sigma a_m x^m$, and we obtain

$$a_{m+1} = \frac{n^2-\frac{1}{4}-m(m+1)}{2(m+1)^2}\,a_m,$$

whence

$$\phi = a_0\left\{1 + (n^2-\tfrac{1}{4})\frac{x}{2} + \frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})}{(2\,!)^2}\left(\frac{x}{2}\right)^2 + \frac{(n^2-\frac{1}{4})(n^2-\frac{9}{4})(n^2-\frac{25}{4})}{(3\,!)^2}\left(\frac{x}{2}\right)^3 + \dots\dots\right\} \dots (30),$$

$= a_0 P$ (say).

Putting $\psi = \Sigma \alpha_m x^m$ we obtain from (29)

$$\alpha_{m+1} = \frac{n^2-\frac{1}{4}-m(m+1)}{2(m+1)^2}\,\alpha_m - \frac{2}{m+1}\,a_{m+1},$$

whence

$$\psi = a_0 P - 2a_0 \Sigma_1^\infty \left(1 + \tfrac{1}{2} + \dots \frac{1}{m}\right)(n^2 - \tfrac{1}{4})(n^2 - \tfrac{9}{4})\dots$$

$$\dots\{n^2 - (m - \tfrac{1}{2})^2\} \frac{x^m}{2^m (m\,!)^2} \dots(31),$$

and therefore $$b_n = a_0 P \log \frac{q-1}{16(q+1)} + \psi \dots\dots(32).$$

By § 273 (56)

$$4nqb_n = (2n-1)\,b_{n-1} + (2n+1)\,b_{n+1}.$$

Hence when q is nearly equal to unity, this equation may be written,

$$4nb_n = (2n-1)\,b_{n-1} + (2n+1)\,b_{n+1},$$

the solution of which is

$$b_n = C + C'\left(1 + \tfrac{1}{3} + \dots \frac{1}{2n-1}\right)$$

$$= C + C' f(n) \dots\dots(33).$$

Therefore $b_1 = C + C'$,

$$2b_0 = C.$$

Therefore $$b_n = 2b_0 + (b_1 - 2b_0) f(n) \dots\dots (34).$$

Now $$2b_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{d\theta}{\sqrt{(q - \cos\theta)}}$$

$$= \frac{2}{\pi\sqrt{(q+1)}}\int_0^{\pi} \frac{d\phi}{(1 - k^2\cos^2\phi)^{\frac{1}{2}}}, \quad k^2 = \frac{2}{q+1}$$

$$= \frac{4}{\pi\sqrt{(q+1)}}\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1 - k^2\sin^2\phi)^{\frac{1}{2}}}$$

$$= \frac{4}{\pi\sqrt{(q+1)}} \log\frac{4}{k'}$$

$$= -\frac{\sqrt{2}}{\pi}\log\frac{q-1}{16(q+1)}$$

approximately. Also

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos\theta\, d\theta}{(q - \cos\theta)^{\frac{1}{2}}}$$

$$= -\frac{4}{\pi}\sqrt{(q+1)}\int_0^{\frac{1}{2}\pi}(1 - k^2\sin^2\phi)^{\frac{1}{2}}\,d\phi + 2b_0 q$$

$$= -4\sqrt{2}/\pi + 2b_0,$$

approximately; whence (34) becomes,

$$b_n = -\frac{\sqrt{2}}{\pi}\log\frac{q-1}{16(q+1)} - \frac{4\sqrt{2}}{\pi} f(n) \dots\dots (35).$$

Since x is very small we may put $x=0$ in the expressions for $\phi(x)$ and $\psi(x)$, and (32) becomes,

$$b_n = a_0 \log \frac{q-1}{16(q+1)} + \alpha_0 \ldots\ldots\ldots\ldots\ldots\ldots (36).$$

Comparing (35) and (36) we obtain

$$a_0 = -\sqrt{2}/\pi,$$
$$\alpha_0 = -4\sqrt{2} f(n)/\pi.$$

If σ_n denote the sum of the reciprocals of the first n natural numbers,

$$f(n) = \sigma_{2n} - \tfrac{1}{2}\sigma_n.$$

But it is shown in Boole's *Finite Differences, 2nd edition*, p. 93 that $\quad \sigma_n = \cdot 577215 + \log n + 2^{-1}n^{-1} - 12^{-1}n^{-2},$

whence

$$f(n) = \cdot 288607 + \log 2n - \tfrac{1}{2}\log n + (48)^{-1}n^{-2} \ldots\ldots (37),$$

and we obtain from (32)

$$b_n = \sqrt{2}\pi^{-1}\{1 + \tfrac{1}{2}(n^2 - \tfrac{1}{4})x\}\{\log 16(2+x)/x - 4f(n)\} + \sqrt{2}\pi^{-1}x(n^2 - \tfrac{1}{4}) \ldots (38),$$

and therefore from (26)

$$c_n = \sqrt{2}\pi^{-1}[2/x - (n^2 - \tfrac{1}{4})\{\log 16(2+x)/x - 4f(n)\} - n^2 - \tfrac{3}{4}] \ldots (39).$$

Hence we finally obtain

$$S_n = [2/x - (n^2 - \tfrac{1}{4})\{\log 16(2+x)/x - 4f(n)\} - n^2 - \tfrac{3}{4}]/2\pi(\varpi a)^{\frac{3}{2}} \ldots (40),$$

where $\quad x = q - 1 = \{(\varpi - a)^2 + \zeta^2\}/2\varpi a.$

318. We can now complete the solution of the equations of § 316. At the surface of the ring $\varpi - a = e\cos\phi$, $\zeta = e\sin\phi$, whence $\quad x = e^2/2a^2, \ (2+x)/x = 4a^2/e^2,$

and therefore

$$S_n = [4a^2/e^2 - (n^2 - \tfrac{1}{4})\{\log 64\, a^2/e^2 - 4f(n)\} - n^2 - \tfrac{3}{4}]/2\pi a^3,$$

and

$$2S_0 = (4a^2/e^2 + \tfrac{1}{2}\log 8a/e - \tfrac{3}{4})/2\pi a^3.$$

Substituting the values of S_n and S_0 in (23) and (24) we obtain

$$\dot{\mathfrak{z}} = m(\log 8a/e - 1)/2\pi a \ldots\ldots\ldots\ldots\ldots\ldots\ldots (41),$$
$$\Phi = -m(4a^2/e^2 - \tfrac{3}{2}\log 8a/e + \tfrac{5}{4})/4\pi a^2$$

or since $\quad m = \pi\omega e^2,$

$$\Phi = -\omega + \tfrac{3}{8}\omega e^2 a^{-2}(\log 8a/e - \tfrac{5}{6}) \ldots\ldots\ldots\ldots (42).$$

Equation (41) gives the velocity of translation of the ring, and agrees with the expression previously obtained in (5). The angular velocity of the liquid at the surface of the ring is given by (42).

319. In order to obtain the equations for determining the small oscillations, we must substitute the value of S_n in (19) and (25). Putting

$$L = m\,\{\log 8a/e - 2f(n) - \tfrac{1}{2}\}/2\pi a^2 \quad\text{.......... (43)},$$

we shall after reduction obtain

$$\dot{\alpha}_n = -n^2 L\gamma_n, \quad \dot{\gamma}_n = (n^2 - 1)\,L\alpha_n \quad\text{.............. (44)},$$

the solution of which is

$$\alpha_n = A\cos\{Ln\sqrt{(n^2-1)}\,t + \beta\},$$

$$\gamma_n = An^{-1}\sqrt{(n^2-1)}\sin\{Ln\sqrt{(n^2-1)}\,t + \beta\}.$$

These equations show that a circular vortex ring is stable for all displacements of its central line, and that the period of oscillation is $2\pi/Ln\sqrt{(n^2-1)}$.

Now e is a small quantity and therefore if n is not very large, $\log 8a/e$ will be large compared with $2f(n) + \frac{1}{2}$, and the period of oscillation is approximately equal to

$$4\pi^2 a^2/mn\sqrt{(n^2-1)}\log 8a/e.$$

But if n is so large that ne is comparable with a, we must substitute for $f(n)$ its value from (37), and we obtain

$$L = m\,(\log 2a/ne - 1{\cdot}0772)/2\pi a^2.$$

Since n is large, we may write n^2 for $n^2 - 1$, hence if $l = 2\pi a/n$ the period of vibration becomes

$$l^2\,(\log l/\pi e - 1{\cdot}0772)^{-1}\,(\pi\omega e^2)^{-1}.$$

The transverse vibrations of a rectilinear vortex have been investigated by Sir W. Thomson[1], who finds that when l/e is large, the period of oscillation is equal to

$$l^2\,(\log l/\pi e - {\cdot}3272)^{-1}\,(\pi\omega e^2)^{-1},$$

which approximately agrees with the preceding expression.

If the displacement had been represented by the equations

$$\varpi' = a + \alpha_n\cos n\psi + \beta_n\sin n\psi, \quad z' = \zeta + \gamma_n\cos n\psi + \delta_n\sin n\psi,$$

it could have been shown in a similar manner that β_n, δ_n satisfy the same equations as α_n, γ_n.

[1] *Phil. Mag.* Sep. 1880, p. 167.

Linked Vortices.

320. The subject of linked vortices has been elaborately discussed by Prof. J. J. Thomson in Part III. of his *Motion of Vortex Rings*, to which the reader must be referred for complete information on the subject. In the present section we shall confine ourselves to the discussion of the simple case of two vortices of equal strengths.

We have shown in § 291, that when two rectilinear vortices are situated at a distance from one another which is large in comparison with the linear dimensions of the cross sections of either, their cross sections will retain an approximately circular form; and the vortices will revolve about their common centre of inertia with angular velocity $(m+m')/\pi d^2$, where m, m' are the strengths of the vortices and d is the shortest distance between them. Hence if the motion is steady the angular velocity must be approximately constant, and therefore d must be constant.

If we consider two linked vortices whose shortest distance is small in comparison with the radii of their apertures, but large in comparison with the linear dimensions of the cross sections of either of them, the action of one vortex upon the other so far as it affects the form of the cross section of the other, will be approximately the same as that of two rectilinear vortices. Hence in order that the cross sections of the two linked vortices may retain an approximately circular form, we must suppose them linked in such a manner that the above conditions are satisfied. When the vortices are of equal strengths, this may be effected by supposing them wound round an anchor ring, the radius of whose cross section is small compared with the radius of its aperture, in such a manner that there are always portions of the two vortices at opposite extremities of a diameter of the cross section of the anchor ring. If we wind a piece of string n times round a curtain ring, and tie the ends together; and then wind another piece of string n times round the ring in the same direction as the first, so that the shortest distance between every point on one of the strings from the other string is a diameter of the cross section of the ring, and tie the ends of the latter together; we shall have an exact representation of the manner of linking.

It is also evident that the number of windings must not exceed a certain number which depends on the dimensions of the cross sections of the vortices and the anchor ring, and also upon the radius of the latter, otherwise the shortest distance might not be the diameter of the cross section of the ring. Moreover one or more vortices twisted round an anchor ring a great number of times would approximate to a vortex sheet, and the motion would be unstable.

321. We shall now consider the small oscillations of two equal vortices wound r times round an anchor ring.

Let the equations of the two vortices when undisturbed, be

$$\varpi = a + \tfrac{1}{2}d\cos r\theta, \qquad z = \zeta + \tfrac{1}{2}d\cos r\theta,$$
$$\varpi' = a - \tfrac{1}{2}d\cos r\theta, \qquad z' = \zeta - \tfrac{1}{2}d\cos r\theta,$$

and let these equations when the vortices are disturbed, be

$$\varpi = a + \Sigma\alpha_n\cos n\theta, \qquad z = \zeta + \Sigma\gamma_n\cos n\theta,$$
$$\varpi' = a + \Sigma\alpha'_n\cos n\theta, \qquad z' = \zeta + \Sigma\gamma'_n\cos n\theta.$$

Also let A_n, S_n be the quantities denoted by these letters in § 316, due to the action of the first vortex upon itself; and let A'_n, S'_n be the values of these quantities, due to the action of the first vortex on the second.

From (18) it follows that the velocity in the direction of the radius due to the first vortex at a point on the second vortex, consists of a series of terms of the type

$$\tfrac{1}{2}ma\left[S'_1\gamma'_n + \tfrac{1}{2}\gamma_n\{(n-1)S'_{n+1} - (n+1)S'_{n-1}\}\right]\cos n\theta \ldots(45).$$

The value of S'_n is given by (40) in terms of x; in the present case x is approximately equal to $d^2/2a^2$, where d is the diameter of the cross section of the anchor ring on which the vortices lie and which is therefore a small quantity. Also if we suppose that n is not sufficiently large for $f(n)$ to be comparable with $\log 8a/d$, it follows that if the largest terms only are retained, the above expression for the velocity

$$= m(4\pi a^2)^{-1}\left[(4a^2/d^2 - \tfrac{3}{2}\log 8a/d)\gamma'_n\right.$$
$$\left. - \{4a^2/d^2 + 2(n^2 - \tfrac{3}{4})\log 8a/d\}\gamma_n\right]\cos n\theta \ldots(46).$$

From (43) and (44) it follows that the velocity along the radius vector due to the action of the second vortex upon itself

$$= -mn^2(2\pi a^2)^{-1}\gamma'_n\cos n\theta\log 8a/e \ldots\ldots\ldots\ldots (47).$$

Since we suppose that e is small compared with α_n, it follows from (15) that the velocity of the second vortex along the radius vector is approximately equal to $\dot{\alpha}'_n \cos n\theta$, whence equating these values of the radial velocity we obtain

$$\dot{\alpha}'_n = m\,[(4a^2/d^2 - \tfrac{3}{2}\log 8a/d - 2n^2 \log 8a/e)\,\gamma'_n$$
$$- \{4a^2/d^2 + 2\,(n^2 - \tfrac{3}{4})\log 8a/d\}\,\gamma_n]/4\pi a^2 \ldots (48).$$

From (12) it follows that the portion of the velocity parallel to z of the second vortex, which is due to the first is,

$$w = \tfrac{1}{2}m\,[2A'_0 a^2 - a\varpi A'_1$$
$$+ \{2S'_n a + \tfrac{1}{2}a\,[(n-1)\,S'_{n-1} - (n+1)\,S'_{n+1}]\}\,\alpha_n \cos n\theta]$$
$$+ \tfrac{1}{2}ma^2 \frac{d}{da}\,\{S'_n - \tfrac{1}{2}\,(S'_{n+1} + S'_{n-1})\}\,\alpha_n \cos n\theta.$$

Now at the second vortex

$$2A'_0 a^2 - a\varpi A'_1 = a^2\,(2S'_0 - S'_1) - S'_1 a\alpha'_n \cos n\theta$$
$$+ a^2 \frac{d}{d\varpi}\,(2S'_0 - S'_1)\,\alpha'_n \cos n\theta,$$

also $$2S'_0 - S'_1 = (\log 8a/d - 1)/\pi a^3,$$

and $$d\,(2S'_0 - S'_1)/d\varpi = -\tfrac{3}{2}\pi^{-1}\,a^{-4}\log 8a/d,$$

retaining the most important term only; whence the value of w approximately is,

$$w = m\,(\log 8a/d - 1)/2\pi a$$
$$- m\,(4\pi a^2)^{-1}\,[(4a^2/d^2 + \tfrac{3}{2}\log 8a/d)\,\alpha'_n$$
$$+ \{4a^2/d^2 + 2\,(n^2 - \tfrac{1}{4})\log 8a/d\}\,\alpha_n]\cos n\theta.$$

By (41) and (44) the velocity parallel to z of the second vortex due to itself is

$$m\,(\log 8a/e - 1)/2\pi a + m\,(2\pi a^2)^{-1}\,(n^2 - 1)\,\alpha'_n \cos n\theta \log 8a/e.$$

The resultant velocity parallel to z of the second vortex is the sum of these two expressions; but by (21) this velocity is also equal to

$$\dot{\zeta} + \dot{\gamma}'_n \cos n\theta,$$

whence equating coefficients in these two expressions, we obtain

$$\dot{\zeta} = m\,(\log 64a^2/de - 2)/2\pi a,$$

$$\dot{\gamma}'_n = m\,[2\,\{2a^2/d^2 + (n^2 - \tfrac{1}{4})\log 8a/d\}\,\alpha_n$$
$$- \{4a^2/d^2 + \tfrac{3}{2}\log 8a/d - 2\,(n^2 - 1)\log 8a/e\}\,\alpha'_n]/4\pi a^2 \ldots (49).$$

322. Let

$$L = m\,(4a^2/d^2 - \tfrac{3}{2}\log 8a/d - 2n^2 \log 8a/e)/4\pi a^2,$$
$$M = m\,\{2a^2/d^2 + (n^2 - \tfrac{3}{4}) \log 8a/d\}/2\pi a^2,$$
$$P = m\,\{4a^2/d^2 + \tfrac{3}{2}\log 8a/d - 2\,(n^2 - 1) \log 8a/e\}/4\pi a^2,$$
$$Q = m\,\{2a^2/d^2 + (n^2 - \tfrac{1}{4}) \log 8a/d\}/2\pi a^2,$$

and we obtain from (48) and (49)

$$\dot{\alpha}'_n = L\gamma'_n - M\gamma_n, \quad \dot{\gamma}'_n = Q\alpha_n - P\alpha'_n.$$

Similarly it can be shown that

$$\dot{\alpha}_n = L\gamma_n - M\gamma'_n, \quad \dot{\gamma}_n = Q\alpha'_n - P\alpha_n,$$

whence

$$\dot{\alpha}' - \dot{\alpha}_n = (L + M)\,(\gamma'_n - \gamma_n), \quad \dot{\gamma}'_n - \dot{\gamma}_n = -\,(P + Q)\,(\alpha'_n - \alpha_n),$$

therefore

$$\left.\begin{aligned} \alpha' - \alpha_n &= 2A \cos(\mu t + \epsilon) \\ \gamma'_n - \gamma_n &= -2A\mu\,(L + M)^{-1} \sin(\mu t + \epsilon) \end{aligned}\right\} \qquad (50),$$

where

$$\begin{aligned} \mu^2 &= (L + M)\,(P + Q) \\ &= (m/4\pi a^2)^2\,\{8a^2/d^2 + (2n^2 - 3) \log 8a/d - 2n^2 \log 8a/e\} \\ &\qquad \times \{8a^2/d^2 + (2n^2 + 1) \log 8a/d - 2\,(n^2 - 1) \log 8a/e\} \ldots (51). \end{aligned}$$

Again,

$$\dot{\alpha}'_n + \dot{\alpha}_n = (L - M)\,(\gamma'_n + \gamma_n), \quad \dot{\gamma}_n + \dot{\gamma}_n = -\,(P - Q)\,(\gamma'_n + \gamma_n),$$

whence

$$\left.\begin{aligned} \alpha'_n + \alpha_n &= 2B \cos(\nu t + \epsilon') \\ \gamma'_n + \gamma_n &= -\,2B\nu\,(L - M)^{-1} \sin(\nu t + \epsilon') \end{aligned}\right\} \ldots\ldots\ldots (52),$$

where

$$\begin{aligned} \nu^2 &= (L - M)\,(P - Q) \\ &= (m/2\pi a^2)^2\,(\log 64a^2/de)^2\, n^2\,(n^2 - 1). \end{aligned}$$

Therefore

$$\begin{aligned} \nu &= m\,(2\pi a^2)^{-1}\, n\,(n^2 - 1)^{\frac{1}{2}} \log 64a^2/de \ldots\ldots\ldots\ldots (53) \\ &= n\,(n^2 - 1)^{\frac{1}{2}}\, V/a \end{aligned}$$

nearly if V be the velocity of translation of the vortex; we therefore finally obtain

$$\left.\begin{aligned} \alpha'_n &= A \cos(\mu t + \epsilon) + B \cos(\nu t + \epsilon') \\ \alpha_n &= -\,A \cos(\mu t + \epsilon) + B \cos(\nu t + \epsilon') \\ \gamma'_n &= -\,A\mu\,(L + M)^{-1} \sin(\mu t + \epsilon) - B\nu (L - M)^{-1} \sin(\nu t + \epsilon') \\ \gamma_n &= A\mu\,(L + M)^{-1} \sin(\mu t + \epsilon) - B\nu\,(L - M)^{-1} \sin(\nu t + \epsilon') \end{aligned}\right\} \ldots (54).$$

Equations (51) and (53) show that μ and ν are both real, and therefore the steady motion we have been considering is both possible and stable; also μ is much greater than ν, and therefore the motion consists of a quick vibration whose period is $2\pi/\mu$ and a slow vibration whose period is $2\pi/\nu$.

323. In the problem we have been considering, we have supposed the vortices wound r times round an anchor ring, and that the equation of its projection on the plane of the ring during the disturbed motion is

$$\varpi = a + \Sigma\alpha_n \cos n\theta.$$

Of the terms α_n, the quantity α_r is the most important, since its maximum value is $\frac{1}{2}d$; the other terms denote small sinuosities and are very much less than α_r. Now (54) shows that if any of the quantities α_n... are initially zero, and the vortex suffers no external disturbance, they will remain zero throughout the motion, and the motion of the vortex will be given by (54), r being written for n; also if the rings are initially placed so that

$$\alpha_r = -\alpha'_r = \tfrac{1}{2}d; \quad \gamma_r = -\gamma'_r = \tfrac{1}{2}d,$$

we see from (52) that $B = D = 0$, and therefore the slow vibrations will not be excited unless the ring suffers some external disturbance.

324. The preceding investigation shows that two vortices of *equal strengths* linked round an anchor ring in the manner described in § 320 are stable; Prof. Thomson has also shown that two linked vortices may be stable when their strengths are *unequal*, but the manner of linking is not the same in the two cases.

When the vortices are of unequal strengths m, m' they must be linked in the following manner[1]:

"Describe an anchor ring whose mean radius of aperture is a, and the radius of whose transverse section is $m'd/(m+m')$; then the central line of vortex core of the vortex of strength m will always lie on the surface of this anchor ring. Describe another anchor ring with the same circular axis, and the same radius of aperture as the first, but with a transverse section of radius $md/(m+m')$; then the central line of vortex core of the vortex ring, whose strength is m', will always lie on the surface of this anchor

[1] *Motion of Vortex Rings*, p. 88.

ring; and will be so situated with respect to the first vortex ring that if we take a transverse section of the anchor ring, and if C be the common centre of the two circular sections, P and Q the points where the central lines of the vortex rings cut the plane of section, then P, C, Q will be in one straight line and C will be between P and Q. If we imagine the circular axis of the anchor rings to move forward with a certain velocity V, and the circular axes of the vortex rings to rotate round it with a certain angular velocity which depends upon their strengths and dimensions, we shall get a complete representation of the motion."

325. A similar method might be employed to investigate the steady motion of a number of linked vortices, but if the number of vortices exceed a certain limit the steady motion will be unstable. For if we suppose for simplicity that the vortices are of equal strengths, and are linked round an anchor ring, the system will approximate to a vortex sheet if the number of vortices be large; and since the cross section of the anchor ring is small compared with the radius of its aperture, such a vortex sheet may be approximately regarded as a cylindrical vortex sheet, and we have shown in the previous chapter that such a vortex sheet is unstable. For the purpose of investigating this question, Prof. Thomson has examined the stability of a number of rectilinear vortices of equal strengths arranged at equal distances round the circumference of a circle, and he finds that the steady motion of six or any less number of vortices is stable, but that seven vortices are unstable; whence it is inferred that if less than seven vortices are linked round an anchor ring so as to cut any cross section in the angular points of a regular polygon, the system is stable, but if there are more than six vortices the system is unstable[1].

Vortex Rings of Finite Section[2].

326. In the preceding investigations we have regarded the cross section of the ring as indefinitely small, and have taken no account of what goes on inside the ring; we shall now suppose that the cross section though small in comparison with the aperture of the ring is finite, and we shall investigate the motion of the rotationally moving liquid of which the ring is composed.

[1] For the motion of vortices in a gas, see Chree, *Mess. Math.* vol. XVII. p. 105.

[2] Hicks, *Phil. Trans.* 1884 and 1885.

For the sake of greater generality we shall suppose that the liquid constituting the ring is of different density from the liquid surrounding it, and that in the surrounding liquid there is a circulation additional to that produced by the filaments of which the ring is composed; but it will be assumed that the pressure at a distance from the ring is sufficient to prevent the formation of a hollow, and the conditions for this will be found.

Let ρ be the density of the outside liquid, μ its circulation; σ the density of the liquid constituting the ring, μ' the circulation due to it.

Outside the ring, Stokes' current function satisfies the equation

$$\frac{d^2\psi}{dz^2} + \frac{d^2\psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\psi}{d\varpi} = 0 \qquad (55).$$

Inside the ring, ψ satisfies the equation

$$\frac{d^2\psi'}{dz^2} + \frac{d^2\psi'}{d\varpi^2} - \frac{1}{\varpi}\frac{d\psi'}{d\varpi} = 2\omega\varpi \qquad (56).$$

In order to obtain the solutions of these equations in a suitable form, it will be necessary to employ the toroidal functions whose properties have been discussed in Chapter XII., and we shall begin by considering the steady motion of the ring.

By § 79 the vorticity at any point of the ring is proportional to ω/ϖ; hence by (33) of § 38 when the motion is steady the vorticity is a function of the current function. Now before it is possible to discuss the properties of any vortex ring it is necessary to know its vorticity, and we shall suppose in the present investigation that the vorticity is constant. This requires that $\omega/\varpi = \text{const.} = \frac{1}{2}M$, whence (56) becomes

$$\frac{d^2\psi'}{dz^2} + \frac{d^2\psi'}{d\varpi^2} - \frac{1}{\varpi}\frac{d\psi'}{d\varpi} = M\varpi^2 \qquad (57).$$

We may also suppose that the ring is at rest, provided we impress upon the whole liquid a translatory velocity equal and opposite to the velocity V of the ring; whence the proper solutions of (55) and (57) may be respectively written,

$$\psi = -\tfrac{1}{2}V\varpi^2 + (2b)^{-\frac{1}{2}}(C+c)^{-\frac{1}{2}}\Sigma_0^\infty A_n R_n (b/k)^{n+\frac{1}{2}} \cos n\xi \ldots (58),$$

and

$$\psi' = \tfrac{1}{8}M\varpi^4 + (2b^5)^{-\frac{1}{2}}(C+c)^{-\frac{1}{2}}\Sigma_0^\infty B_n T_n (k/b)^{n-\frac{1}{2}} \cos n\xi \ldots\ldots (59).$$

If the ring contained a hollow space, it would be necessary to

include terms of the form $D_n R_n (b/k)^{n+\frac{1}{2}} \cos n\xi$ in the expression for ψ'; but as we suppose that the pressure is sufficient to prevent the formation of a hollow, ψ' cannot contain any terms of this form.

327. We are not at liberty to assume that the cross section of the ring is an exact circle in steady motion; but when the cross section is small compared with the aperture, it can be represented by an equation of the form

$$k = b\,(1 + \beta_1 \cos\xi + \beta_2 \cos 2\xi + \ldots\ldots) \ldots\ldots\ldots (60),$$

where b is a small quantity and β_n is another small quantity of the order b^n; and our object will be to obtain an approximate solution of the problem upon this assumption, which as we shall presently see is justified by the result. We shall make the further assumption, which is also justified by the result, that A_n and B_n are each quantities of the order b^n; and for a first approximation we shall retain quantities of the first order in calculating ψ, which will render it unnecessary to carry the approximation farther than the term involving $\cos\xi$; but in calculating ψ' it will be necessary to carry the approximation as far as $\cos 2\xi$, and to include in the coefficients of these terms quantities of the third order.

328. Putting $C = \cosh\eta$, $S = \sinh\eta$, $c = \cos\xi$, we have shown that

$$J = (C + c)/a, \quad \varpi = aS/(C + c) \ldots\ldots\ldots\ldots (61),$$

also by § 280 if p and q are the velocities perpendicular to the surfaces η and ξ measured in the directions shown in the figure of that section,

$$p = J\varpi^{-1}\, d\psi/d\xi, \quad q = J\varpi^{-1}\, d\psi/d\eta \ldots\ldots\ldots\ldots (62).$$

Since μ' is the circulation due to the ring

$$\begin{aligned}
\mu' &= 2\iint \omega\, d\sigma = M\iint \varpi J^{-2}\, d\eta\, d\xi, \\
&= Ma^3 \iint (C + c)^{-3}\, S\, d\eta\, d\xi, \\
&= -\tfrac{1}{2} Ma^3 \int_{-\pi}^{\pi} (C + c)^{-2}\, d\xi, \\
&= -4M\pi a^3 b^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (63),
\end{aligned}$$

terms of the fourth order being omitted. Also since μ is the circulation outside the ring, it follows from (70) of § 281, writing $A_n b^n 2^{-\frac{1}{2}}$ for A_n, that

$$\mu = -\pi a^{-1} (A_0 - A_1 b + A_2 b^2 -) \ldots\ldots\ldots\ldots\ldots (64).$$

329. We are now in a position to calculate ψ. From (61) we obtain

$$\varpi^2 = a^2(1-k^2)^2/(1+k^2+2k\cos\xi),$$
$$= a^2\{1-4k\cos\xi+2k^2(1+3\cos 2\xi)-\} \quad\ldots (65).$$

Also

$$(C+c)^{-\frac{1}{2}} = (2k)^{\frac{1}{2}}/(1+k^2+2k\cos\xi)^{\frac{1}{2}},$$
$$= (2k)^{\frac{1}{2}}\{1+\tfrac{1}{4}k^2-(k+\tfrac{3}{8}k^3)\cos\xi+\tfrac{3}{4}k^2\cos 2\xi-\}\ldots(66),$$

in which we have retained quantities of the third order in the coefficient of $\cos\xi$, as they will hereafter be required. Whence to the first order

$$\psi = -\tfrac{1}{2}Va^2(1-4k\cos\xi)+(1-k\cos\xi)\{A_0R_0+A_1R_1(b/k)\cos\xi\}(67).$$

The value of ψ must be constant at the surface, if therefore we substitute for k its value from (60), the coefficient of $\cos\xi$ must vanish. Now by § 283

$$\left.\begin{aligned} R_0 &= -\tfrac{1}{2}(L-2)-\tfrac{1}{8}k^2(L+1) \\ R_1 &= \tfrac{1}{2}-\tfrac{3}{4}k^2(L-\tfrac{1}{2}) \end{aligned}\right\}\ldots\ldots\ldots\ldots(68).$$

Therefore
$$\left.\begin{aligned} dR_0/dk &= (2k)^{-1}-\tfrac{1}{4}k(L+\tfrac{1}{2}) \\ dR_1/dk &= -\tfrac{3}{2}k(L-1) \end{aligned}\right\}\ldots\ldots\ldots\ldots(69).$$

Therefore at the surface

$$\psi = -\tfrac{1}{2}Va^2(1-4b\cos\xi)+(1-b\cos\xi)$$
$$\times\{-\tfrac{1}{2}A_0(L-2)+\tfrac{1}{2}A_0\beta_1\cos\xi+\tfrac{1}{2}A_1\cos\xi\}.$$

Equating the coefficient of $\cos\xi$ to zero, we obtain

$$2Va^2+\tfrac{1}{2}A_0(L-2+\beta_1/b)+\tfrac{1}{2}A_1/b=0\ldots\ldots\ldots(69a),$$

which shows that A_1 is of the first order; therefore from (64)

$$A_0 = -\mu a/\pi \quad\ldots\ldots\ldots\ldots(70),$$
$$A_1 = \mu\pi^{-1}ab(L-2+\beta_1/b)-4Va^2b\ldots\ldots\ldots\ldots(71).$$

330. The calculation of ψ' is more difficult, since we must retain terms of the third order. Let $Q=-\mu' a/4\pi$, then by (63), $M=Q/a^4b^2$, and the value of ψ' becomes

$$b^2\psi' = Q\varpi^4/8a^4+(2b)^{-\frac{1}{2}}(C+c)^{-\frac{1}{2}}\Sigma_0^\infty B_nT_n(k/b)^{n-\frac{1}{2}}\cos n\xi\ldots(72).$$

Now $\varpi^4 = a^4\{1+12k^2-8(k+6k^3)\cos\xi+20k^2\cos 2\xi\}$,
also by § 283

$$T_0 = 1+\tfrac{1}{4}k^2,\ T_1=\tfrac{3}{2}(1-\tfrac{1}{8}k^2),\ T_2=\tfrac{15}{8}(1-\tfrac{1}{4}k^2),$$

omitting k^4.

Therefore by (66)

$$(2b)^{-\frac{1}{2}}(C+c)^{-\frac{1}{2}}B_0T_0(b/k)^{\frac{1}{2}}=B_0\{1+\tfrac{1}{4}k^2-(k+\tfrac{3}{8}k^3)\cos\xi+\tfrac{3}{4}k^2\cos 2\xi\}\times(1+\tfrac{1}{4}k^2)$$

$$=B_0\{1+\tfrac{1}{2}k^2-(k+\tfrac{5}{8}k^3)\cos\xi+\tfrac{3}{4}k^2\cos 2\xi\},$$

$$(2b)^{-\frac{1}{2}}(C+c)^{-\frac{1}{2}}B_1T_1(k/b)^{\frac{1}{2}}=\tfrac{3}{2}B_1(1+\tfrac{1}{4}k^2-k\cos\xi+\tfrac{3}{4}k^2\cos 2\xi)\times(k/b)(1-\tfrac{1}{8}k^2)\cos\xi,$$

$$=-\tfrac{3}{2}B_1\{\tfrac{1}{2}k-(1+\tfrac{1}{2}k^2)\cos\xi+\tfrac{1}{2}k\cos 2\xi\}\,k/b.$$

$$(2b)^{-\frac{1}{2}}(C+c)^{-\frac{1}{2}}B_2T_2(k/b)^{\frac{3}{2}}=\tfrac{15}{8}B_2(1-k\cos\xi)(k/b)^2\cos 2\xi,$$

$$=-\tfrac{15}{8}B_2(\tfrac{1}{2}k\cos\xi-\cos 2\xi)(k/b)^2.$$

Collecting the terms and putting for brevity

$$G=Q+B_0-3B_1/2b,\quad H=15B_2/4b^2-3B_1/2b+\tfrac{3}{2}B_0+5Q,$$

we obtain

$$b^2\psi'=\tfrac{1}{8}Q+B_0+(Q+\tfrac{1}{2}G)k^2-\{Gk+(\tfrac{1}{4}H+\tfrac{1}{4}G+\tfrac{9}{2}Q)k^3\}\cos\xi$$
$$+\tfrac{1}{2}Hk^2\cos 2\xi\ldots\ldots\ldots\ldots(73).$$

In order to obtain the surface value of ψ', we must substitute the value of k from (60) in (73).

The first two terms

$$=\tfrac{1}{8}Q+B_0+b^2(Q+\tfrac{1}{2}G)(1+2\beta_1\cos\xi).$$

The next term

$$=-\{Gb+b^3(\tfrac{1}{4}H+\tfrac{1}{4}G+\tfrac{9}{2}Q)+\tfrac{1}{2}Gb\beta_2\}\cos\xi-\tfrac{1}{2}Gb\beta_1(1+\cos 2\xi).$$

The last term

$$=\tfrac{1}{2}Hb^2(\cos 2\xi+\beta_1\cos\xi).$$

Adding and equating the coefficients of $\cos\xi$ and $\cos 2\xi$ to zero, we obtain

$$-Gb-(\tfrac{1}{4}H+\tfrac{1}{4}G+\tfrac{9}{2}Q)b^3-\tfrac{1}{2}Gb\beta_2+\tfrac{1}{2}Hb^2\beta_1+(2Q+G)b^2\beta_1=0\ldots(74),$$

and

$$-\tfrac{1}{2}Gb\beta_1+\tfrac{1}{2}Hb^2=0\ldots\ldots\ldots\ldots(75).$$

From (74) it follows that to the lowest order

$$G=0,$$

whence from (75) $\qquad H=0.$

From these equations it appears that G and H are quantities of the fourth order at least.

Substituting in (73) we obtain

$$b^2\psi'=\tfrac{1}{8}Q+B_0+Qk^2-\tfrac{9}{2}Qk^3\cos\xi\ldots\ldots\ldots\ldots(76).$$

This is the approximate value of ψ' inside the ring to the first order of small quantities.

331. We can now determine the value of β_1. Since the normal velocity must be zero at the surface of the ring, the boundary condition is

$$q\frac{dF}{d\xi} - p\frac{dF}{d\eta} = 0,$$

which by (62) becomes

$$\frac{d\psi'}{dk}\frac{dF}{d\xi} - \frac{d\psi'}{d\xi}\frac{dF}{dk} = 0,$$

where $$F = b(1 + \beta_1 \cos\xi) - k = 0.$$

Therefore

$$(4Qk - 27Qk^2\cos\xi)\, b\beta_1 \sin\xi - 9Qk^3 \sin\xi = 0,$$

whence to the lowest order

$$\beta_1 = \tfrac{9}{4}b \qquad\qquad (77).$$

332. We must now calculate the pressure. Inside the ring the equations of steady motion are

$$\frac{1}{\sigma}\frac{dp'}{dz} + \tfrac{1}{2}\frac{dq^2}{dz} - 2u\omega = 0,$$

$$\frac{1}{\sigma}\frac{dp'}{d\varpi} + \tfrac{1}{2}\frac{dq^2}{d\varpi} + 2w\omega = 0,$$

where q is the resultant velocity, whence remembering that $\varpi u = -d\psi'/dz$, $\varpi w = d\psi'/d\varpi$, $2\omega = M\varpi$, we obtain

$$p'/\sigma = E - \tfrac{1}{2}q^2 + M\psi' \qquad\qquad (78),$$

where E is a constant. Outside the ring the pressure is determined by the equation

$$p = \Pi - \tfrac{1}{2}\rho q^2,$$

where Π is the pressure at infinity. Now

$$q^2 = \frac{J^2}{\varpi^2}\left\{\left(\frac{d\psi}{d\eta}\right)^2 + \left(\frac{d\psi}{d\xi}\right)^2\right\},$$

$$= \frac{J^2k^2}{\varpi^2}\left\{\left(\frac{d\psi}{dk}\right)^2 + \frac{1}{k^2}\left(\frac{d\psi}{d\xi}\right)^2\right\},$$

and $$Jk/\varpi = (1 + 4k\cos\xi)/2a^2,$$

approximately; also from (76)

$$\frac{d\psi'}{dk} = \frac{Q}{b^2}\left(2k - \frac{27}{2}k^2\cos\xi\right),$$

therefore $$\frac{Jk}{\varpi}\frac{d\psi'}{dk} = \frac{Q}{2a^2b^2}\left(2k - \frac{11}{2}k^2\cos\xi\right),$$

in which we have neglected terms of the first order, since they are small in comparison with b^{-1}. For the same reason $k^{-1}\, d\psi/d\xi$ may be neglected, whence (78) becomes

$$\frac{p'}{\sigma} = E - \frac{Q^2}{8a^4b^4}(2k - \tfrac{11}{2}k^2\cos\xi)^2 + \frac{Q^2}{a^4b^4}(\tfrac{1}{8} + B_0/Q + k^2 - \tfrac{9}{2}k^3\cos\xi).$$

The velocity evidently vanishes when $k = 0$, whence if P be the pressure along the critical circle

$$P/\sigma = E + Q(\tfrac{1}{8}Q + B_0)/a^4b^4,$$

whence at the surface (omitting terms of zero order),

$$\frac{p'}{\sigma} = \frac{P}{\sigma} + \frac{Q^2}{2a^4b^2}\{1 + (2\beta_1 - \tfrac{7}{2}b)\cos\xi\} \quad \text{............ (79).}$$

From (67) we obtain

$$\frac{d\psi}{dk} = (2Va^2 - A_0R_0)\cos\xi + (1 - k\cos\xi)(A_0/2k - A_1b/2k^2 \,.\, \cos\xi).$$

Therefore at the surface

$$\frac{Jk}{\varpi}\frac{d\psi}{dk}$$

$$= \frac{1}{2a^2}(1 + 4b\cos\xi)\,[A_0/2b + \{2Va^2 + \tfrac{1}{2}A_0(L-3)$$

$$- \tfrac{1}{2}A_0\beta_1/b - A_1/2b\}\cos\xi]$$

$$= \frac{1}{2a^2}[A_0/2b + \{4Va^2 + A_0(L - \tfrac{1}{2})\}\cos\xi],$$

by (71).

Therefore

$$q^2 = \frac{1}{4a^4b^4}[\tfrac{1}{4}A_0^2 + A_0b\{4Va^2 + A_0(L - \tfrac{1}{2})\}\cos\xi],$$

hence $$\frac{p}{\rho} = \frac{\Pi}{\rho} - \frac{A_0}{8a^4b^2}[\tfrac{1}{4}A_0 + b\{4Va^2 + A_0(L - \tfrac{1}{2})\}\cos\xi] \quad \text{... (80).}$$

Since the pressure must be the same on either side of the surface of separation, we obtain by equating the values of p, p' given by (79) and (80),

$$P + Q^2\sigma/2a^4b^2 = \Pi - A_0^2\rho/32a^4b^2 \quad \text{..............(81),}$$

$$Q^2(2\beta_1/b - \tfrac{7}{2})\sigma = -\tfrac{1}{4}A_0\rho\{4Va^2 + A_0(L - \tfrac{1}{2})\} \quad \text{......(82).}$$

Putting for Q and A_0 their values, these become

$$P = \Pi - \frac{\mu^2\rho + \mu'^2\sigma}{32a^2b^2} \quad \text{............................ (83),}$$

$$V = \frac{\mu}{4\pi a}(L - \tfrac{1}{2}) + \frac{\mu'^2\sigma}{16\pi a\mu\rho}\left(\frac{2\beta_1}{b} - \frac{7}{2}\right) \quad \text{...... (84).}$$

333. Equation (83) determines the pressure along the critical circle, hence it follows that in order that there should be no hollow, P must never become negative; this requires that

$$\Pi = \text{or} > \frac{\mu^2\rho + \mu'^2\sigma}{32a^2b^4} \quad \text{.................(85).}$$

If therefore the pressure at infinity is slightly less than the above quantity, a hollow will begin to form about the critical circle.

The velocity of translation of the ring is determined by (84). Let e be the radius of the circle which approximates most nearly to the cross section of the ring, then

$$e = a \operatorname{cosech} \eta = 2ab,$$

therefore $$L = \log 4/b = \log 8a/e$$

and there are three cases to be considered.

(i) Let $\mu = \mu'$, $\rho = \sigma$, $m = \frac{1}{2}\mu$, where m is the strength of the vortex; then substituting the value of β_1 from (77), (84) becomes

$$V = \frac{m}{2\pi a}(\log 8a/e - \tfrac{1}{4}) \quad \text{.................(86),}$$

which gives the velocity of translation of a ring of the same density as the liquid, when there is no additional circulation.

This expression does not agree with that obtained for the velocity in (5) and (41), but it must be recollected that since e is small compared with a, $\log 8a/e$ is large compared with $\frac{1}{4}$, and therefore the difference between the two expressions is small. The present procedure, although more complicated, gives a perfectly accurate result to the order of approximation adopted, and the next term in the value of V is of the first order of small quantities.

(ii) Let there be a ring-shaped hollow round which circulation takes place.

The conditions for the existence of such a hollow are that p should be zero at the surface, and also that $\mu' = \sigma = 0$; hence from (83) and (84) or directly from (80) we obtain

$$\left.\begin{aligned} V &= \frac{m}{2\pi a}(\log 8a/e - \tfrac{1}{2}) \\ \Pi &= \mu^2\rho/32a^2b^2 \end{aligned}\right\} \quad \text{.................(87).}$$

(iii) It is also possible to have a ring-shaped mass of liquid relatively at rest, surrounded by liquid in a state of cyclic irrotational motion. In this case the surface of separation will be a surface of discontinuity along which slipping takes place, which possesses the properties of a vortex sheet. The condition for this is that $\mu' = 0$, whence the liquid constituting the ring is relatively at rest, and it moves forward like a rigid body with a velocity V which is given by (87). In order that the liquid should be continuous at the surface of separation, it is necessary that

$$\Pi = \text{or} > \mu^2\rho/32a^2b^2.$$

It can be proved that in cases (ii) and (iii) the value of β_1 is of the second order of small quantities, see Appendix.

Fluted Vibrations of a Vortex Ring.

334. We have shown in § 319, that a vortex of small cross section is stable with respect to a deformation of its central line; we shall now investigate the effect of a deformation of its cross section, such that the disturbance consists of trains of waves whose crests are circles which are parallel to the critical circle. These vibrations may be called *fluted vibrations.*

Instead of adopting a procedure analogous to that employed in § 288 for investigating the corresponding vibrations of a rectilinear vortex, it will be more convenient to use complex quantities and throw away the imaginary part[1]; we shall therefore suppose that the cross section of the ring at time t is represented by an equation of the form

$$k = b + b\Sigma\beta_n\epsilon^{\iota n\xi + \iota\lambda t} \quad\text{.......................(88).}$$

In the beginning of the disturbed motion the β's will be small compared with b, except β_1 whose mean value we have already shown to be equal to $\frac{9}{4}b$; we may therefore in considering the variations of β_n regard the cross section of the ring as an exact circle in steady motion; but the value of β_1 thus obtained can only be regarded as a first approximation, and a more accurate result would be obtained by going to a second approximation.

[1] The employment of complex quantities was suggested to me by Mr A. E. H. Love.

We shall not however consider in detail this latter point[1], but proceed at once to determine the value of β_n on the supposition that the cross section is an exact circle in steady motion.

335. Dropping the suffix n, (88) so far as β_n is concerned may be written

$$k = b + b\beta\epsilon^{\iota n\xi + \iota\lambda t} \qquad (89),$$

also let the current function due to the disturbed motion be

$$\chi = (2b)^{-\frac{1}{2}}(C+c)^{-\frac{1}{2}} AR_n (b/k)^{n+\frac{1}{2}} \epsilon^{\iota n\xi + \iota\lambda t}$$

outside, and

$$\chi' = (2b)^{-\frac{1}{2}}(C+c)^{-\frac{1}{2}} BT_n (k/b)^{n-\frac{1}{2}} \epsilon^{\iota n\xi + \iota\lambda t}$$

inside. The coefficients A and B will be small complex constants of the order β; also by (66), $(C+c)^{-\frac{1}{2}} = (2k)^{\frac{1}{2}}$ approximately; also since R_n, T_n are respectively multiplied by small quantities they may approximately be regarded as constants; we may therefore write

$$\chi = A(b/k)^n \epsilon^{\iota n\xi + \iota\lambda t} \qquad (90),$$

$$\chi' = B(k/b)^n \epsilon^{\iota n\xi + \iota\lambda t} \qquad (91).$$

336. If p and q are the velocities perpendicular to the surfaces η and ξ measured in the directions shown in the figure to § 280, the boundary condition is

$$\frac{dF}{dt} + J\left(q\frac{dF}{d\xi} - p\frac{dF}{d\eta}\right) = 0.$$

Since $J^{-1} = 2ak$, we obtain from (89)

$$\iota b(2ab\lambda\beta + qn\beta)\epsilon^{\iota n\xi + \iota\lambda t} - pk = 0 \qquad (92).$$

Outside the ring

$$q = \frac{J}{\varpi}\frac{d}{d\eta}(\psi + \chi) = \frac{1}{2a^2k}\frac{d}{d\eta}(\psi + \chi).$$

Since q is multiplied by a small quantity, the term $d\chi/d\eta$ may be neglected; also from (67) the principal term in $(2a^2k)^{-1} d\psi/d\eta$ is $\mu/4\pi ak$, which at the surface of the ring is approximately equal to U the tangential velocity just outside the ring in steady motion; we may therefore in the small terms put $q = U$. Also

$$p = \frac{1}{2a^2k}\frac{d}{d\xi}(\psi + \chi).$$

[1] A similar question arises in connection with linked vortices, which Prof. J. J. Thomson has investigated by carrying the approximation to the second order. This would be very laborious in the present case.

Now $\frac{1}{2}a^{-2}k^{-1}d\psi/d\xi$ is very nearly equal to the normal velocity of the ring in steady motion, and may therefore be neglected (since the ring is supposed to be at rest in consequence of its velocity of translation having been reversed); we may therefore put

$$pk = \frac{1}{2a^2}\frac{d\chi}{d\xi} = \frac{A\iota n}{2a^2}\epsilon^{\iota n\xi+\iota\lambda t},$$

whence (92) becomes

$$A = 2a^2b\beta\,(U + 2ab\lambda/n) \quad\text{.....................(93).}$$

If U' be the tangential velocity just inside the ring, it can be shown in the same manner that

$$B = 2a^2b\beta\,(U' + 2ab\lambda/n) \quad\text{..................(94).}$$

337. We must in the next place determine the velocity potential due to the disturbed motion.

Since the disturbed motion is irrotational and acyclic, its velocity potential at any point P is equal to the flow along any path joining P with the origin. Let this path be the curve $\xi = 0$ from $\eta = 0$ to $\eta = \eta$, and the curve $\eta = \eta$ from $\xi = 0$ to $\xi = \xi$. Then

$$\phi = -\int_0^\eta J^{-1}pd\eta + \int_0^\xi J^{-1}qd\xi$$

$$= -\int_0^\eta \left(\frac{1}{\varpi}\frac{d\chi}{d\xi}\right)_{\xi=0} d\eta + \int_0^\xi \frac{1}{\varpi}\frac{d\chi}{d\eta}\,d\xi.$$

Substituting the value of χ from (90), we obtain

$$\left(\frac{1}{\varpi}\frac{d\chi}{d\xi}\right)_{\xi=0} = \frac{1+\cosh\eta}{a\sinh\eta}A\iota n\epsilon^{n\eta+\iota\lambda t}$$

$$= A\iota na^{-1}(b/k)^n\,(1 + 2k + \ldots)\,\epsilon^{\iota\lambda t},$$

whence keeping only the largest terms we obtain, since k is small,

$$-\int_0^\eta J^{-1}pd\eta = -A\iota a^{-1}(b/k)^n\,\epsilon^{\iota\lambda t}.$$

Also

$$\int J^{-1}qd\xi = Ana^{-1}(b/k)^n\int_0^\xi \epsilon^{\iota n\xi+\iota\lambda t}\,d\xi$$

$$= -A\iota a^{-1}(b/k)^n\,\epsilon^{\iota n\xi+\iota\lambda t} + A\iota a^{-1}(b/k)^n\,\epsilon^{\iota\lambda t},$$

whence $$\phi = -A\iota a^{-1}(b/k)^n\,\epsilon^{\iota n\xi+\iota\lambda t} \quad\text{..................(95).}$$

Similarly it can be shown that

$$\phi' = B\iota a^{-1}(k/b)^n\,\epsilon^{\iota n\xi+\iota\lambda t} \quad\text{.....................(96).}$$

338. Putting for a moment $k = b + \delta k$, the pressure outside the ring is determined by the equation

$$\frac{p}{\rho} = \text{const.} - \dot{\phi} - \tfrac{1}{2}\left(U + \frac{dU}{dk}\delta k + \frac{J}{\varpi}\frac{d\chi}{d\eta}\right)^2 + \text{terms of 2nd order.}$$

Hence if δp, $\delta p'$ be the increments of the pressure p due to the disturbed motion just outside and just inside the vortex,

$$\frac{\delta p}{\rho} = -\dot{\phi} - U\left(\frac{dU}{dk}\delta k - \frac{1}{2a^2}\frac{d\chi}{dk}\right).$$

Now from (67) it follows that to a first approximation

$$U = \mu/4\pi ak,$$

therefore $$dU/dk = -U'k,$$

whence dropping the exponential factor

$$\delta p/\rho = -A\lambda/a + U^2\beta - nUA/2a^2b.$$

Substituting the value of A from (93) we obtain

$$\delta p/\rho = \beta\left\{U^2 - n\left(U + 2ab\lambda/n\right)^2\right\};$$

if therefore we write β_1 and w for $2ab\beta$ and $U/2ab$ respectively, we shall obtain

$$n\delta p/2ab\rho = -\beta_1\left\{\lambda^2 + 2\lambda nw + n(n-1)w^2\right\}\text{.........}(97).$$

Just inside the vortex the pressure p' is

$$\frac{p'}{\sigma} = \text{const.} - \dot{\phi}' - \tfrac{1}{2}\left(U' + \frac{dU'}{dk}\delta k + \frac{J}{\varpi}\frac{d\chi'}{d\eta}\right)^2 + M\left(\psi' + \frac{d\psi'}{dk}\delta k + \chi'\right),$$

whence

$$\frac{\delta p'}{\sigma} = -\dot{\phi}' - U'\left(\frac{dU'}{dk}\delta k - \frac{1}{2a^2}\frac{d\chi'}{dk}\right) + M\left(\frac{d\psi'}{dk}\delta k + \chi'\right).$$

But $M = -\mu'/4\pi a^3b^2$, also from (76) $U' = \mu'k/4\pi ab^2$, therefore

$$dU'/dk = U'/k, \quad Ma^2 = -U'/k;$$

also $$U' = -\tfrac{1}{2}a^{-2}\,d\psi'/dk,$$

whence omitting the exponential factor

$$\begin{aligned}\delta p'/\sigma &= -\dot{\phi}' + U'^2\delta k/k + \tfrac{1}{2}U'a^{-2}\,d\chi'/dk - U'\chi'/ka^2\\ &= B\lambda/a + U'^2\beta + nUB/2a^2b - U'B/a^2b\\ &= \beta\left[U'^2 + (U' + 2ab\lambda/n)\left\{2ab\lambda + (n-2)U'\right\}\right].\end{aligned}$$

Putting $U'/2ab = v$, this becomes

$$n\delta p'/2ab\sigma = \beta_1\left\{\lambda^2 + 2(n-1)\lambda v + n(n-1)v^2\right\}\text{......}(98).$$

Since $\delta p = \delta p'$, we obtain

$$\rho \{\lambda^2 + 2n\lambda w + n(n-1)w^2\} + \sigma \{\lambda^2 + 2(n-1)\lambda v + n(n-1)v^2\} = 0,$$

or writing f for σ/ρ this becomes

$$\lambda^2(1+f) + 2\lambda\{nw + f(n-1)v\} + n(n-1)(w^2 + fv^2) = 0 \ldots (99).$$

In order that the steady motion should be stable it is necessary that both roots of this quadratic should be real.

Referring to § 288 it appears that the period equation (99) is exactly the same as equation (21) of that section with the sign of λ changed. This however does not affect the question of stability; hence the conditions of stability are the same in both cases, as might have been expected, since a circular vortex whose cross section is small compared with its aperture, approximates to a rectilinear vortex. It therefore follows that if there is slipping at the surface of the ring, the steady motion must be unstable.

339. We have shown in § 332 that if the pressure at a great distance from the vortex is less than $(\mu^2\rho + \mu'^2\sigma)/32a^2b^2$ a hollow space must exist within the ring; and that if this pressure is just below this critical value, the hollow must begin to form at the critical circle. The steady motion of a ring in which such a hollow exists, when there is an additional circulation inside the ring, which is always possible when a hollow exists, has been considered by Mr Hicks, and one curious point connected with the investigation is, that it seems probable that under certain circumstances the hollow might slip out of the ring, so that two rings might be formed, one of which consists of a hollow with circulation round it, and the other consists of a rotational core with no additional circulation; but until the subject has been more fully investigated, it cannot be asserted that this state of things could actually take place.

In Mr Hicks' investigation from which the foregoing articles are taken, the more general problem of the fluted vibrations of a vortex when there is a hollow and an additional internal circulation is considered. It should however be noticed that his period equations (63) and (65)[1] do not agree with equation (99) of § 338. Unless therefore some error exists in the analysis of §§ 334—338, his results upon this portion of the subject cannot be regarded as altogether free from doubt.

[1] *Phil. Trans.* 1885.

340. When a hollow exists in the ring, it is possible for it to pulsate as well as to vibrate. The question of pulsations has also been considered by Mr Hicks.

EXAMPLES.

1. Prove that effect of a circular vortex at a great distance from itself, is approximately the same as that of a doublet of strength $\frac{1}{2}mc^2$, where m is the strength of the vortex, and c is its radius.

2. The motion of an incompressible fluid in a spherical vessel at any instant, is such that each spherical stratum rotates like a rigid shell, the rectangular components of its velocity being ω_1, ω_2, ω_3, these quantities varying from stratum to stratum; show that if each element of fluid is attracted towards the centre with a force whose intensity per unit of mass is

$$(\omega_1 x + \omega_2 y + \omega_3 z)\left(x\frac{d\omega_1}{dr} + y\frac{d\omega_2}{dr} + z\frac{d\omega_3}{dr}\right) + \frac{dV}{dr},$$

where V is any function of the coordinates, the motion of the fluid will be steady; and find the pressure at any point.

3. If p_1 be the period of the quick vibrations when two vortices of equal strengths are linked once through each other, and p_2 when they are linked twice through one another; show that

$$\frac{7}{p_1} - \frac{1}{p_2} = \frac{6m}{\pi^2 d^2},$$

and prove also that the period of the vibrations gets longer, as the complexity of linking increases.

4. Prove that the current function due to a fine circular vortex of radius c and strength m, may be expressed in the form

$$m\varpi a \int_0^\infty \epsilon^{\pm\lambda(z-z')} J_1(\lambda\varpi) J_1(\lambda c)\, d\lambda,$$

the upper or lower sign being taken according as $z - z'$ is negative or positive.

5. A closed vessel bounded by two coaxial circular cylinders of radii a and b respectively and of lengths $2h$, with plane ends perpendicular to the axis, is filled with liquid in rotational motion, the vorticity being uniform, and the planes of the vortex filaments parallel to the axis. Show that when the motion is steady, the current function is of the form

$$\psi = \zeta (\varpi^2 - a^2)(\varpi^2 - b^2) - \zeta r \Sigma L_n \left\{ \frac{I_1(n\varpi)}{I_1(na)} - \frac{K_1(n\varpi)}{K_1(na)} \right\} \frac{\cos nz}{\cos nh},$$

where the summation extends to all values of n given by the equation

$$I_1(na) K_1(nb) = I_1(nb) K_1(na).$$

6. If Θ, Θ' are the velocities of the liquid surrounding a thin circular vortex ring of strength m, at two points in the plane of the ring each of which is the inverse of the other with respect to the radius of the ring, and whose distances from the centre of the ring are R, R', where $R > R'$; prove that

$$\Theta \sqrt{R} + \Theta' \sqrt{R'} = \frac{2m}{\pi} \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(R - R' \sin^2 \theta)^{\frac{1}{2}}}.$$

CHAPTER XV.

ON THE MOTION OF A LIQUID ELLIPSOID UNDER THE INFLUENCE OF ITS OWN ATTRACTION.

341. In the year 1738 the Academy of Sciences at Paris offered a prize for an essay on the subject of the theory of the tides. The authors of four essays received prizes, viz. Maclaurin, Euler, Daniel Bernoulli and Cavalleri. The essay of Maclaurin is chiefly of importance, owing to his having proved that when a mass of liquid is rotating as a rigid body about a fixed axis under the influence of its own attraction, a possible form of the free surface is a planetary ellipsoid, whose polar axis coincides with the axis of rotation. In 1834 Jacobi discovered that under the same conditions, another possible form of the free surface is an ellipsoid with three unequal axes, whose least axis coincides with the axis of rotation. The researches of Dirichlet, Dedekind and Riemann have shown, that the ellipsoidal form is a possible form of the free surface, when the liquid does *not* rotate as a rigid body. The discussion of these different ellipsoids forms the subject of the present chapter.

342. We shall commence by obtaining the general equations of motion of a mass of liquid, which rotates about its centre of inertia under the influence of its own attraction, in such a manner that its free surface always remains an ellipsoid with variable axes. The motion of the liquid is supposed to be rotational, but the molecular rotation is assumed to be independent of the positions of individual elements of liquid, and it will be shown that the consequence of this assumption is, that the component velocities at any point of the liquid are linear functions

of the coordinates of that point. We shall first of all show that the particular kind of motion under consideration, may be generated from rest by means of the following three operations, which are supposed to take place instantaneously one after the other[1].

(i) Let an ellipsoidal case whose axes are a, b, c be filled with liquid and frozen, and then set in rotation with component angular velocities ξ, η, ζ about the principal axes.

(ii) Let the liquid be melted, and let additional angular velocities Ω_1, Ω_2, Ω_3 be impressed on the case.

(iii) Let the case be removed, and by means of a suitable impulsive pressure applied to the free surface, let the axes be made to vary with velocities $\dot{a}$, $\dot{b}$, $\dot{c}$.

343. Let x, y, z be the coordinates of an element of liquid referred to the principal axes; u, v, w the component velocities of the element parallel to the axes; U, V, W the component velocities relative to the axes; and ω_1, ω_2, ω_3 the angular velocities of the axes about themselves. Then

$$\omega_1 = \Omega_1 + \xi, \quad \omega_2 = \Omega_2 + \eta, \quad \omega_3 = \Omega_3 + \zeta.$$

The kinematical condition to be satisfied at the free surface is

$$\frac{dF}{dt} + U\frac{dF}{dx} + V\frac{dF}{dy} + W\frac{dF}{dz} = 0 \quad \text{..............(1)},$$

where $$F = (x/a)^2 + (y/b)^2 + (z/c)^2 - 1 = 0,$$

and $$U = u + \omega_3 y - \omega_2 z,$$

$$V = v + \omega_1 z - \omega_3 x,$$

$$W = w + \omega_2 x - \omega_1 y.$$

Equation (1) can be satisfied by assuming

$$u = l_1 x + m_1 y + n_1 z,$$

$$v = l_2 x + m_2 y + n_2 z,$$

$$w = l_3 x + m_3 y + n_3 z,$$

where l_1, m_1, &c. are independent of x, y and z.

Substituting in (1) and equating coefficients of powers and products of x, y, z to zero, we obtain

[1] Greenhill, *Proc. Camb. Phil. Soc.* vol. IV. p. 4.

$$l_1 = \dot{a}/a, \quad m_2 = \dot{b}/b, \quad n_3 = \dot{c}/c,$$
$$(n_2 + \omega_1)\, c^2 + (m_3 - \omega_1)\, b^2 = 0,$$
$$(l_3 + \omega_2)\, a^2 + (n_1 - \omega_2)\, c^2 = 0,$$
$$(m_1 + \omega_3)\, b^2 + (l_2 - \omega_3)\, a^2 = 0.$$

But from the mode of generation ξ, η, ζ are independent of x, y, z; therefore

$$2\xi = m_3 - n_2, \quad 2\eta = n_1 - l_3, \quad 2\zeta = l_2 - m_1.$$

Hence the nine coefficients are completely determined, and we finally obtain

$$\left.\begin{aligned}
u &= \frac{\dot{a}x}{a} + \frac{\omega_3 (a^2 - b^2) - 2a^2\zeta}{a^2 + b^2}\, y + \frac{\omega_2 (c^2 - a^2) + 2a^2\eta}{c^2 + a^2}\, z \\
v &= \frac{\dot{b}y}{b} + \frac{\omega_1 (b^2 - c^2) - 2b^2\xi}{b^2 + c^2}\, z + \frac{\omega_3 (a^2 - b^2) + 2b^2\zeta}{a^2 + b^2}\, x \\
w &= \frac{\dot{c}z}{c} + \frac{\omega_2 (c^2 - a^2) - 2c^2\eta}{c^2 + a^2}\, x + \frac{\omega_1 (b^2 - c^2) + 2c^2\xi}{b^2 + c^2}\, y
\end{aligned}\right\} \ldots (2).$$

These expressions obviously satisfy the equation of continuity, since on account of the constancy of volume

$$\dot{a}/a + \dot{b}/b + \dot{c}/c = 0.$$

344. By § 23 (4) the general equations for the pressure referred to moving axes are

$$\frac{1}{\rho}\frac{dp}{dx} - X + \frac{du}{dt} - v\omega_3 + w\omega_2 + U\frac{du}{dx} + V\frac{du}{dy} + W\frac{du}{dz} = 0 \ldots (3),$$

with two similar equations; and by eliminating the pressure and potential, the equations for molecular rotation will be found to be

$$\frac{d\xi}{dt} - \eta\omega_3 + \zeta\omega_2 + U\frac{d\xi}{dx} + V\frac{d\xi}{dy} + W\frac{d\xi}{dz} = \xi\frac{du}{dx} + \eta\frac{du}{dy} + \zeta\frac{du}{dz} \ldots (4),$$

with two similar equations. Substituting the values of u, v, w from (2) in (4) we shall obtain

$$\left.\begin{aligned}
\frac{d}{dt}\left(\frac{\xi}{a}\right) - \frac{2ab}{a^2 + b^2}\Omega_3\left(\frac{\eta}{b}\right) + \frac{2ca}{c^2 + a^2}\Omega_2\left(\frac{\zeta}{c}\right) = 0 \\
\frac{d}{dt}\left(\frac{\eta}{b}\right) - \frac{2bc}{b^2 + c^2}\Omega_1\left(\frac{\zeta}{c}\right) + \frac{2ab}{a^2 + b^2}\Omega_3\left(\frac{\xi}{a}\right) = 0 \\
\frac{d}{dt}\left(\frac{\zeta}{c}\right) - \frac{2ca}{c^2 + a^2}\Omega_2\left(\frac{\xi}{a}\right) + \frac{2bc}{b^2 + c^2}\Omega_1\left(\frac{\eta}{b}\right) = 0
\end{aligned}\right\} \ldots\ldots (5).$$

If h_1, h_2, h_3 are the component angular momenta about the axes

$$\left.\begin{aligned}
h_1 &= \rho\iiint(wy - vz)\,dx\,dy\,dz,\\
&= \frac{M}{5(b^2+c^2)}\{(b^2-c^2)^2\omega_1 + 4b^2c^2\xi\}\\
h_2 &= \frac{M}{5(c^2+a^2)}\{(c^2-a^2)^2\omega_2 + 4c^2a^2\eta\}\\
h_3 &= \frac{M}{5(a^2+b^2)}\{(a^2-b^2)^2\omega_3 + 4a^2b^2\zeta\}
\end{aligned}\right\}\text{ ....(6)},$$

where M is the mass of the liquid; and the dynamical equations for rotation are

$$\left.\begin{aligned}
\frac{dh_1}{dt} - h_2\omega_3 + h_3\omega_2 &= 0\\
\frac{dh_2}{dt} - h_3\omega_1 + h_1\omega_3 &= 0\\
\frac{dh_3}{dt} - h_1\omega_2 + h_2\omega_1 &= 0
\end{aligned}\right\}\text{ ....(7).}$$

If we now introduce the six new quantities u, v, w, u', v', w' employed by Riemann, such that

$$\left.\begin{aligned}
u + u' &= \omega_1, & v + v' &= \omega_2, & w + w' &= \omega_3,\\
u - u' &= \frac{2bc\Omega_1}{b^2+c^2}, & v - v' &= \frac{2ca\Omega_2}{c^2+a^2}, & w - w' &= \frac{2ab\Omega_3}{a^2+b^2}
\end{aligned}\right\}\text{ ...(8),}$$

we obtain

$$\left.\begin{aligned}
\xi &= \{(b+c)^2u' - (b-c)^2u\}/2bc, \text{ \&c., \&c.}\\
h_1 &= \tfrac{1}{5}M\{(b+c)^2u' + (b-c)^2u\}, \text{ \&c., \&c.}
\end{aligned}\right\}\text{ ....(9).}$$

Substituting these values of ξ, η, ζ, and h_1, h_2, h_3 in (5) and (7), and then multiplying (5) by $\frac{2}{5}Mabc$ and adding to (7), we obtain

$$(b+c)\frac{du'}{dt} + 2u'\frac{d}{dt}(b+c) + (b-c+2a)vw' + (b-c-2a)v'w = 0\text{...(10).}$$

Similarly by subtracting, we obtain

$$(b-c)\frac{du}{dt} + 2u\frac{d}{dt}(b-c) + (b+c-2a)vw + (b+c+2a)v'w' = 0\text{...(11).}$$

Four other equations can respectively be written down by symmetry, and we thus obtain six equations of motion.

345. The three remaining equations can be obtained as follows. The potential of the liquid at an internal point is

$$V = \tfrac{1}{2}(Ax^2 + By^2 + Cz^2) - H,$$

where
$$H = \tfrac{3}{4}M\int_0^\infty \frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},$$

$$A = -\frac{2}{a}\frac{dH}{da}.$$

Now if in equations (3) we transpose the terms $\rho^{-1}dp/dx - X$ &c., to the right-hand sides, and then substitute the values of the velocities given by (2), the left-hand side of each equation will be a linear function of x, y, z; moreover if we multiply each equation by dx, dy, dz and add, the right hand side of the resulting equation will be a perfect differential, and therefore the left hand side must be so also. Hence (3) must be of the form

$$\left.\begin{aligned} \frac{1}{\rho}\frac{dp}{dx} + Ax + \alpha x + hy + gz = 0 \\ \frac{1}{\rho}\frac{dp}{dy} + By + hx + \beta y + fz = 0 \\ \frac{1}{\rho}\frac{dp}{dz} + Cz + gx + fy + \gamma z = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(12).$$

The last three terms of these equations are the component accelerations of an element of liquid parallel to the axes; and since there are no external impressed forces, the moments of these accelerations about the coordinate axes must be zero, hence

$$\Sigma m\,\{(gx + fy + \gamma z)\,y - (hx + \beta y + fz)\,z\} = 0,$$

or
$$f\Sigma m\,(y^2 - z^2) = 0,$$

therefore
$$f = 0,$$

similarly $g = 0$, $h = 0$; and (12) reduce to

$$\left.\begin{aligned} \frac{1}{\rho}\frac{dp}{dx} + (A + \alpha)\,x = 0 \\ \frac{1}{\rho}\frac{dp}{dy} + (B + \beta)\,y = 0 \\ \frac{1}{\rho}\frac{dp}{dz} + (C + \gamma)\,z = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(13),$$

where α, β, γ are quantities which are independent of x, y, z, and which will hereafter be determined.

Integrating we obtain

$$p/\rho + \Pi + \tfrac{1}{2}\{(A+\alpha)x^2 + (B+\beta)y^2 + (C+\gamma)z^2\} = 0 \text{...(14)}.$$

Since the external surface is the ellipsoid

$$(x/a)^2 + (y/b)^2 + (z/c)^2 = 1,$$

we must have

$$(A+\alpha)a^2 = (B+\beta)b^2 = (C+\gamma)c^2 = 2\sigma \text{.........(15)},$$

where σ is a function of the time. Hence (14) may be written

$$\frac{p}{\rho} = \varpi + \sigma\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \text{...............(16)}.$$

In order that the external surface should be a free surface, it is necessary that ϖ should vanish, and consequently σ must never become negative.

346. Returning to (13) we see that α is the coefficient of x in the expression for the component acceleration parallel to x of an element of liquid, and therefore

$$\alpha = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) - \frac{w+w'}{a}\{(a-b)w + (a+b)w'\} + \frac{v+v'}{a}\{(c-a)v - (c+a)v'\},$$

$$+\frac{\dot{a}^2}{a^2} - \frac{(w-w')}{a}\{(a-b)w - (a+b)w'\} + \frac{v-v'}{a}\{(c-a)v + (c+a)v'\},$$

$$= \frac{2}{a}\left\{\tfrac{1}{2}\frac{d^2a}{dt^2} - (a-b)w^2 - (a+b)w'^2 - (a-c)v^2 - (a+c)v'^2\right\},$$

whence by (15),

$$\tfrac{1}{2}\frac{d^2a}{dt^2} - (a-c)v^2 - (a+c)v'^2 - (a-b)w^2 - (a+b)w'^2 = \frac{\sigma}{a} - \tfrac{1}{2}Aa \text{ (17)}.$$

Two other symmetrical equations can be obtained; hence, collecting our results, we have the following nine equations;

$$\left.\begin{aligned}
&\tfrac{1}{2}\ddot{a} - (a-c)v^2 - (a+c)v'^2 - (a-b)w^2 - (a+b)w'^2 = \sigma/a - \tfrac{1}{2}Aa\\
&\tfrac{1}{2}\ddot{b} - (b-a)w^2 - (b+a)w'^2 - (b-c)u^2 - (b+c)u'^2 = \sigma/b - \tfrac{1}{2}Bb\\
&\tfrac{1}{2}\ddot{c} - (c-b)u^2 - (c+b)u'^2 - (c-a)v^2 - (c+a)v'^2 = \sigma/c - \tfrac{1}{2}Cc\\
&(b-c)\dot{u} + 2u(\dot{b}-\dot{c}) + (b+c-2a)vw + (b+c+2a)v'w' = 0\\
&(b+c)\dot{u}' + 2u'(\dot{b}+\dot{c}) + (b-c+2a)vw' + (b-c-2a)v'w = 0\\
&(c-a)\dot{v} + 2v(\dot{c}-\dot{a}) + (c+a-2b)wu + (c+a+2b)w'u' = 0\\
&(c+a)\dot{v}' + 2v'(\dot{c}+\dot{a}) + (c-a+2b)wu' + (c-a-2b)w'u = 0\\
&(a-b)\dot{w} + 2w(\dot{a}-\dot{b}) + (a+b-2c)uv + (a+b+2c)u'v' = 0\\
&(a+b)\dot{w}' + 2w'(\dot{a}+\dot{b}) + (a-b+2c)uv' + (a-b-2c)u'v = 0
\end{aligned}\right\} \text{(18)}.$$

$$abc = \text{const.}$$

These equations were first obtained by Riemann[1]; they furnish ten independent relations between the ten unknown quantities $a, b, c, u, u', v, v', w, w'$ and σ, and are therefore sufficient for the complete solution of the problem.

347. Three first integrals of the above equations can be at once obtained.

Multiplying equations (5) by ξ/a, η/b, ζ/c, and adding, we obtain

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=\text{const.}\qquad\qquad(19),$$

which expresses the fact that the vorticity is constant[2].

Similarly from (7) we obtain

$$h_1^2+h_2^2+h_3^2=\text{const.}\qquad\qquad(20),$$

which expresses the fact that the angular momentum is constant.

The third integral is the equation of energy

$$T+U=\text{const.}\qquad\qquad(21).$$

Since $\qquad \rho\iiint x^2dxdydz=\frac{1}{5}Ma^2,$

and $\qquad \iiint xydxdydz=0,$

we obtain from (2)

$$T=\tfrac{1}{10}\left\{\dot a^2+\dot b^2+\dot c^2+\frac{\omega_1^2(b^2-c^2)^2}{b^2+c^2}+\frac{\omega_2^2(c^2-a^2)^2}{c^2+a^2}+\frac{\omega_3^2(a^2-b^2)^2}{a^2+b^2}\right.$$
$$\left.+\frac{4b^2c^2\xi^2}{b^2+c^2}+\frac{4c^2a^2\eta^2}{c^2+a^2}+\frac{4a^2b^2\xi^2}{a^2+b^2}\right\}\qquad(22).$$

[1] *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, vol. IX.; see also *Proc. Lond. Math. Soc.* vol. XVII. p. 255.

[2] This equation may be shortly proved thus:—

Since ξ, η, ζ are independent of x, y, z, the vortex lines must all be parallel to some diameter r of the ellipsoid. Let l, m, n be the direction cosines of r, dS an element of the plane conjugate to r, and ϵ the angle between r and S.

The condition that the vorticity should be constant requires that

$$\text{const.}=\iint\omega\sin\epsilon\, dS=\omega S\sin\epsilon=\omega Spr^{-1},$$

where p is the perpendicular from the centre on to the tangent plane parallel to the plane S. But, since the volume of the ellipsoid is constant, $Sp=\text{const.}$, therefore $\omega/r=\text{const.}$, or

$$\omega^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)=\text{const.},$$

i.e., $$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=\text{const.}$$

Now[1] $U = \frac{1}{2}\rho \iiint V dxdydz,$

$$= \tfrac{3}{8}M^2 \int_0^\infty \left\{ \frac{1}{5}\left(\frac{a^2}{a^2+\lambda} + \frac{b^2}{b^2+\lambda} + \frac{c^2}{c^2+\lambda} \right) - 1 \right\} \frac{d\lambda}{P},$$

where $P = \sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}$, therefore

$$U = -\tfrac{3}{20}M^2 \int_0^\infty \frac{d\lambda}{P} + \tfrac{3}{20}M^2 \int_0^\infty \lambda \frac{d}{d\lambda}\left(\frac{1}{P}\right) d\lambda.$$

Integrating the last term by parts we obtain

$$U = -\tfrac{2}{5}M\pi\rho abc \int_0^\infty \frac{d\lambda}{P} \qquad \text{.................(23).}$$

Motion of an Ellipsoid of Revolution[2].

348. Let us now apply the preceding equations to determine the motion when the free surface is an ellipsoid of revolution, which is rotating about its polar axis. Let the density of the liquid be unity, and let $a = b$; and let ω_1, ω_2, ξ, η, Ω_1, Ω_2, Ω_3 be each zero; then $\omega_3 = \zeta$, $w = w' = \frac{1}{2}\zeta$.

From the last of equations (5) we obtain

$$\frac{d}{dt}\left(\frac{\zeta}{c}\right) = 0,$$

therefore $\qquad \zeta/c = \zeta_0/c_0,$

where the suffix denotes the initial values of the quantities.

Let $R^3 = a^2c$, and let us introduce two new variables θ and ρ, such that

$$\theta = R^2/a^2 = c/R$$

and $\qquad \rho = \zeta/(2\pi)^{\frac{1}{2}} = \zeta_0\, c/c_0\,(2\pi)^{\frac{1}{2}} = \rho_0\theta/\alpha,$

where α is the initial value of θ. From the first and third of equations (18) we obtain

$$-\tfrac{1}{2}\ddot{\theta} + \frac{3\dot{\theta}^2}{4\theta} - 2\pi\rho^2\theta = \frac{2\sigma\theta^2}{R^2} - A\theta,$$

$$\ddot{\theta} = \frac{\sigma}{R^2\theta} - \tfrac{1}{2}C\theta.$$

1 Maxwell's *Electricity*, vol. I. art. 85.
2 Dirichlet, *Crelle*, vol. LVIII. p. 209.

Eliminating $\dot{\theta}$ and σ, and remembering that $A + \frac{1}{2}C = 2\pi$, we obtain

$$\frac{\sigma}{R^2}\left(2\theta + \frac{1}{\theta^2}\right) = 2\pi(1-\rho^2) + \frac{3\dot{\theta}^2}{4\theta^2},$$

$$2\left(2+\frac{1}{\theta^3}\right)\ddot{\theta} - \frac{3\dot{\theta}^2}{\theta^4} + 8\pi\left(\frac{\rho_0}{a}\right)^2 = 4\left(\frac{A}{\theta^2} - C\theta\right).$$

Putting $$F(\theta) = \int_0^\infty \frac{d\lambda}{(1+\theta\lambda)(1+\lambda/\theta^2)^{\frac{1}{2}}},$$

then $$F'(\theta) = \left(\frac{1}{\theta^3} - 1\right)\int_0^\infty \frac{\lambda d\lambda}{(1+\theta\lambda)^2(1+\lambda/\theta^2)^{\frac{3}{2}}},$$

and the left-hand side of the last equation can be shown to be equal to $8\pi F'(\theta)$; integrating this equation we obtain

$$\left(2+\frac{1}{\theta^3}\right)\dot{\theta}^2 + 8\pi\left\{\left(\frac{\rho_0}{a}\right)^2\theta - F(\theta)\right\} = \text{const.} = 8\pi K,$$

which is the equation of energy. Hence the equations of motion finally become

$$\left.\begin{aligned}\frac{\sigma}{R^2}\left(2\theta + \frac{1}{\theta^2}\right) &= 2\pi(1-\rho^2) + \frac{3\dot{\theta}^2}{4\theta^2}\\ 2\left(2+\frac{1}{\theta^3}\right)\ddot{\theta} - \frac{3\dot{\theta}^2}{\theta^4} + 8\pi\left\{\frac{\rho_0^2}{a^2} - F'(\theta)\right\} &= 0\\ \left(2+\frac{1}{\theta^3}\right)\dot{\theta}^2 + 8\pi\left\{\left(\frac{\rho_0}{a}\right)^2\theta - F(\theta)\right\} &= 8\pi K\end{aligned}\right\} \quad \ldots\ldots(24).$$

These are Dirichlet's equations for the motion of an ellipsoid of revolution.

349. Since the remainder of the present investigation depends upon the properties of the function $F(\theta)$, when θ is positive, it will be convenient to trace the curve $y = F(x)$. Now

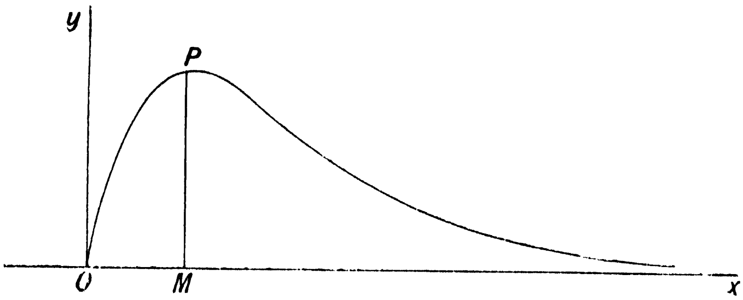
$$F(\theta) = 2\theta^{-1}(\theta^{-3}-1)^{-\frac{1}{2}}\tan^{-1}(\theta^{-3}-1)^{\frac{1}{2}}, \quad \theta < 1;$$

and

$$F(\theta) = \theta^{-1}(1-\theta^{-3})^{-\frac{1}{2}}\log\frac{1+(1-\theta^{-3})^{\frac{1}{2}}}{1-(1-\theta^{-3})^{\frac{1}{2}}}, \quad \theta > 1.$$

Also when $\theta = 0, F(\theta) = 0$, $F'(\theta) = \infty$; when $\theta = 1, F(\theta) = 2$, $F'(\theta) = 0$. When θ increases from 0 to 1, $F(\theta)$ increases from 0 to 2 which is its maximum value; and $F'(\theta)$ is positive and diminishes from ∞ to 0.

When θ increases from 1 to ∞, $F(\theta)$ diminishes from 2 to 0; also $F'(\theta)$ is always negative and vanishes when $\theta=\infty$. Hence the axis of x is an asymptote. The form of the curve is shown in the figure.



350. Let us first suppose that the motion is irrotational, in which case $\rho=\rho_0=0$; also that initially $\dot\theta=0$.

Equations (24) now become

$$\frac{\sigma}{R^2}\left(2\theta+\frac{1}{\theta^2}\right)=2\pi+\frac{3\dot\theta^2}{4\theta^2},$$

$$2\left(2+\frac{1}{\theta^3}\right)\ddot\theta-\frac{3\dot\theta^2}{\theta^4}=8\pi F'(\theta),$$

$$\left(2+\frac{1}{\theta^3}\right)\dot\theta^2=8\pi\left\{F(\theta)-F(\alpha)\right\}.$$

From the last equation it follows that $F(\theta)$ must never be less than $F(\alpha)$, throughout the motion. Now if $\alpha=1$, the initial form of the free surface would be spherical; also since $F(\theta)$ is a maximum when $\theta=1$, it follows that $\theta=\alpha=1$ throughout the motion; hence the free surface always remains spherical.

If $\alpha<1$, the initial form of the free surface would be a planetary ellipsoid; also from the figure, it is seen that the equation $F(\theta)=F(\alpha)$ has one real root β which is greater than 1; hence $\dot\theta$ will vanish when $\theta=\beta$, and therefore the free surface will oscillate through a sphere to an ovary ellipsoid, and back again to its original form, the time of a complete oscillation being

$$=(2\pi)^{-\frac{1}{2}}\int_\alpha^\beta\sqrt{\frac{2+\theta^{-3}}{F(\theta)-F(\alpha)}}\,d\theta.$$

The motion is of a similar kind when the initial form of the free surface is an ovary ellipsoid.

351. The general character of the motion is not altered when the motion does not commence from rest, provided the initial value of $\dot{\theta}$ does not exceed a certain limit. If $\dot{\alpha}$ be the initial value of $\dot{\theta}$, the last of equations (24) becomes

$$(2+\theta^{-3})\,\dot{\theta}^2 = (2+\alpha^{-3})\,\dot{\alpha}^2 + 8\pi\,\{F(\theta)-F(\alpha)\}\ldots\ldots(25).$$

Let $$(2+\alpha^{-3})\,\dot{\alpha}^2 - 8\pi F(\alpha) = 8\pi k,$$

and (25) becomes

$$(2+\theta^{-3})\,\dot{\theta}^2 = 8\pi\,\{F(\theta)+k\}.$$

In order that $\dot{\theta}$ may vanish it is necessary that k should be negative, in which case we may put $k = -F(\gamma)$; hence the ellipsoid will oscillate between the values $\theta=\gamma$, $\theta=\gamma'$, where γ, γ' are the two real roots of the equation $F(\gamma)=F(\theta)$. But if k is positive θ will indefinitely increase or indefinitely diminish with the time according as $\dot{\alpha}$ is positive or negative. In the former case the ellipsoid will gradually become elongated to an indefinite extent, and in the latter case will become indefinitely flattened.

In the foregoing cases σ is always positive, and therefore the motion can take place without the aid of an external pressure.

352. We must now consider the case in which there is molecular rotation.

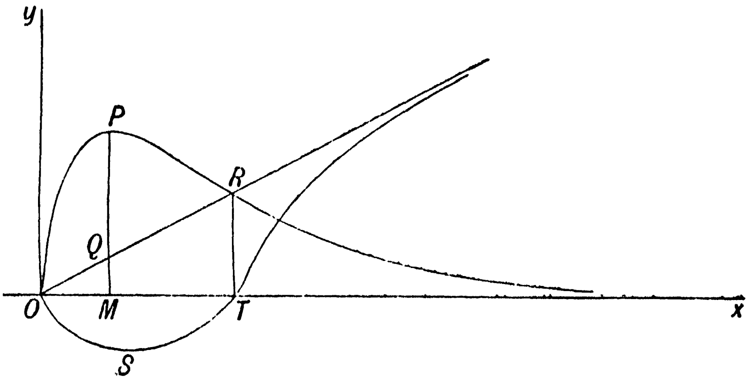
Let δ be a quantity defined by the equation

$$F'(\delta) = (\rho_0/\alpha)^2,$$

then since $F'(\delta)$ is positive, δ must lie between 0 and 1; also let

$$\psi(\theta) = \theta F'(\delta) - F(\theta).$$

The character of the motion depends on the properties of the curve $y=\psi(x)$, which we shall now investigate.



In the figure let OPR and OST be the positive branches of the curves $y=F(x)$ and $y=\psi(x)$ respectively; and let OR be the

straight line $y = xF'(\delta)$. The ordinate of the curve $y = \psi(x)$ will evidently be equal to $QM - PM = -PQ$, and will therefore be negative so long as $x < OT$; also since $\psi'(\delta) = 0$, $\psi(x)$ will be numerically greatest when $x = \delta$, and its value will be negative. When $x > OT$, $\psi(x)$ is positive, and the straight line $y = xF'(\delta)$ is an asymptote to the curve.

353. Putting
$$(2 + \alpha^{-3})\dot{\alpha}^2 = 8\pi k,$$
and remembering that $\rho = \rho_0\theta/\alpha$, (24) may be written

$$\left.\begin{aligned} \frac{\sigma}{R^2}\left(2\theta + \frac{1}{\theta^2}\right) &= 2\pi\,\{1 - \theta^2 F'(\delta)\} + \frac{3\dot{\theta}^2}{4\theta^2} \\ 2\left(2 + \frac{1}{\theta^3}\right)\ddot{\theta} - \frac{3\dot{\theta}^2}{\theta^4} + 8\pi\psi'(\theta) &= 0 \\ \left(2 + \frac{1}{\theta^3}\right)\dot{\theta}^2 + 8\pi\psi(\theta) &= 8\pi\,\{\psi(\alpha) + k\} \end{aligned}\right\} \quad \ldots\ldots\ldots(26).$$

From the last equation it follows that during the whole motion $\psi(\alpha) + k - \psi(\theta)$ can never become negative. Since $\psi(\delta)$ is the greatest negative value that $\psi(\theta)$ can have, there are three cases to be considered according as

(i) $\qquad \psi(\alpha) + k = \psi(\delta)$,

(ii) $\qquad 0 > \psi(\alpha) + k > \psi(\delta)$,

(iii) $\qquad \psi(\alpha) + k > 0$.

Case (i). The equation of condition may be written

$$k = \psi(\delta) - \psi(\alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(27).$$

Now k is always positive, and the right-hand side of (27) is always negative unless $\alpha = \delta$, when it is zero, hence $\alpha = \delta$, $k = 0$; also since $\psi(\delta) - \psi(\theta)$ must never be negative, it follows that $\theta = \delta$ throughout the motion. Now $\delta < 1$, therefore the ellipsoid must be planetary, and the motion is such that the liquid rotates as a rigid body about the axis of the ellipsoid, with angular velocity

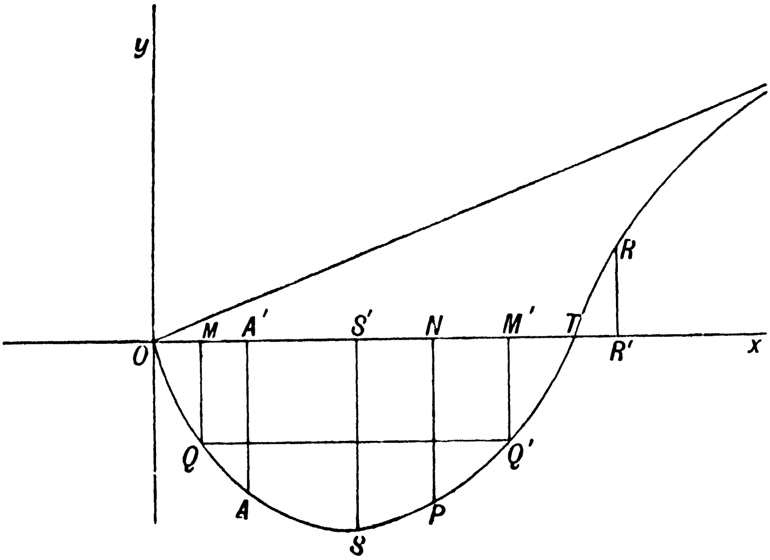
$$\zeta^2 = 2\pi\delta^2 F'(\delta).$$

It will hereafter be shown that the maximum value of the quantity $\theta^2 F'(\theta)$ is $\cdot 2246$, and that the equation $\delta^2 F'(\delta) = \theta^2 F'(\theta)$ has two real roots, δ, δ', both of which are positive and less than unity. Hence for every value of $\zeta^2/2\pi$ which is less than $\cdot 2246$ there are two planetary ellipsoids which are possible forms of the free surface, and which coincide when $\zeta^2/2\pi = \cdot 2246$: also since σ

is always positive the motion can take place without the aid of an external pressure.

This is Maclaurin's ellipsoid, which will be treated in a different manner later on.

Case (ii). Here $\psi(\alpha)+k$ is a negative quantity which is numerically less than $\psi(\delta)$; hence we may put this quantity equal to $\psi(\gamma)$, where $\gamma<\delta$.



In the figure let $OM=\gamma$, $OA'=\alpha$, $OS'=\delta$, $ON=\theta$, $OM'=\gamma'$, where QQ' is parallel to Ox. Then

$$AA'-k=QM;$$

therefore $\gamma<\alpha$; also since $\psi(\alpha)+k-\psi(\theta)$, that is $\psi(\gamma)-\psi(\theta)$, must be always positive,

$$PN>QM.$$

Now the equation $\psi(\gamma)-\psi(\theta)=0$ has evidently two real roots lying between zero and OT, viz. $\theta=\gamma$, $\theta=\gamma'$; hence the ellipsoid will oscillate in such a manner that θ must always lie between γ and γ', and the time of a complete oscillation is

$$(2\pi)^{-\frac{1}{2}}\int_{\gamma'}^{\gamma}\sqrt{\frac{2+\theta^{-3}}{\psi(\gamma)-\psi(\theta)}}\,d\theta\ldots\ldots\ldots\ldots\ldots(28).$$

From the first of equations (26) it follows that the pressure will not remain positive, unless $\theta^2F'(\delta)$ never becomes greater than unity throughout the motion, hence $\gamma'^2F'(\delta)$ must never be greater than unity. Also since

$$\zeta^2=2\pi(\rho_0/\alpha)^2\theta^2=2\pi F'(\delta)\theta^2\ldots\ldots\ldots\ldots(29),$$

this condition requires that $\zeta^2/2\pi$ should never be greater than unity.

Since

$$(2+\alpha^{-3})\,\dot{\alpha}^2/8\pi = \psi(\gamma) - \psi(\alpha)$$
$$= AA' - QM,$$

it follows that we must have

$$\dot{\alpha}^2 < \frac{8\pi\alpha^3}{(2\alpha^3+1)}\{\psi(\gamma) - \psi(\alpha)\} \quad\ldots\ldots\ldots\ldots\ldots(30).$$

If the conditions (29) and (30) are not satisfied, an external pressure will be necessary in order to maintain the ellipsoidal form of the free surface.

Case (iii). In this case $\psi(\alpha) + k$ is always positive, if therefore we put it equal to $\psi(\epsilon)$ where $\epsilon = OR'$, we must have $OR' > OT$. The last of equations (26) becomes

$$(2+\theta^{-3})\,\dot{\theta}^2 = 8\pi\{\psi(\epsilon) - \psi(\theta)\}.$$

The equation $\psi(\epsilon) - \psi(\theta)$ has only one real root, viz. $\theta = \epsilon$, and therefore the motion can never be of an oscillatory character. If $\dot{\theta}$ be initially positive, then since $\psi(\theta)$ is negative so long as $\theta < OT$, and positive when $\theta > OT$, it follows that the ellipsoid will gradually elongate itself to a limiting form determined by the equation $\theta = \epsilon$. On the other hand if $\dot{\theta}$ be initially negative, the ellipsoid will ultimately become indefinitely flattened.

The possibility of this motion taking place without the aid of an external pressure, depends upon conditions similar to those of the preceding case.

Steady Motion of an Ellipsoid.

354. When a mass of liquid is rotating in a state of steady motion under the influence of its own attraction, the different ellipsoidal forms which its free surface can assume may, as we shall proceed to show, be classified as follows.

(i) *Maclaurin's Ellipsoid,* in which the free surface is a planetary ellipsoid, and the liquid rotates as a rigid body about the polar axis of the ellipsoid. If ρ be the density of the liquid, ζ the angular velocity of the ellipsoid, which in this case is identical

with the molecular rotation, it will be shown that $\zeta^2/4\pi\rho$ must not be greater than $\cdot 1123$, in order that steady motion may be possible, and in this case there are two ellipsoids, which coalesce when $\zeta^2/4\pi\rho = \cdot 1123$.

(ii) *Jacobi's Ellipsoid*, in which the free surface is an ellipsoid with three unequal axes, and the liquid rotates as a rigid body about the least axis. In this case $\zeta^2/4\pi\rho$ must not be greater than $\cdot 0934$ in order that the ellipsoid may be a possible form of the free surface. Hence if $\zeta^2/4\pi\rho < \cdot 0934$ there are three ellipsoidal forms, viz. two planetary ellipsoids, and an ellipsoid with three unequal axes. When $\zeta^2/4\pi\rho = \cdot 0934$, Jacobi's ellipsoid coalesces with the most oblate of the two planetary ellipsoids; and when $\zeta^2/4\pi\rho$ lies between $\cdot 0934$ and $\cdot 1123$ the revolutional form is the only one possible.

(iii) *Dedekind's Ellipsoid*, in which the free surface remains stationary in space, but there is an internal motion of the particles of liquid, due to molecular rotation ζ parallel to the least axis. In this case if a and b are the greatest and mean axes respectively, $a^2b^2\zeta^2/(a^2+b^2)^2\,\pi\rho$ must not be greater than $\cdot 0934$; and when the former quantity is equal to $\cdot 0934$, we must have $a = b$, and Dedekind's ellipsoid coalesces with the most oblate of the two Maclaurin's ellipsoids.

(iv) An ellipsoid, which will be called the *Irrotational Ellipsoid*, in which the axis of rotation is the *mean* axis, and the motion is irrotational. In this case the revolutional form is not possible.

(v) An ellipsoid in which there is molecular rotation ζ, and an independent angular velocity $\zeta + \Omega$ about the axis to which ζ refers. In this case the axis of rotation will be the *mean* or *least* axis according as

$$\frac{\zeta}{\Omega} < \text{or} > \frac{a^2-b^2}{a^2+b^2}\left(1 \pm \frac{2a}{\sqrt{(a^2-b^2)}}\right).$$

When this inequality becomes an equality, the free surface will be an ovary ellipsoid rotating about an equatorial axis. This case includes the four preceding cases.

(vi) *Riemann's Ellipsoid*, in which the ellipsoid rotates about an instantaneous axis lying in a principal plane. This case includes all the preceding cases; if the axis of rotation does not lie in a principal plane steady motion is impossible. It is moreover

impossible for steady motion to exist when the axis of rotation is the greatest axis.

The foregoing propositions might be established by employing Riemann's general equations of motion, but when the rotation takes place about a principal axis, it is simpler to start from first principles[1], and we shall therefore commence with Case v.

355. Let c be the axis of rotation, $\omega_3 = \Omega + \zeta$, then from (2) we obtain

$$u = \frac{a^2 - b^2}{a^2 + b^2}\Omega y - \zeta y, \quad v = \frac{a^2 - b^2}{a^2 + b^2}\Omega x + \zeta x, \quad w = 0 \quad \text{...(31)}.$$

The hydrodynamical equations for the pressure referred to the principal axes of the ellipsoid are therefore

$$\left.\begin{aligned} \frac{1}{\rho}\frac{dp}{dx} + Ax - v(\Omega + \zeta) + U\frac{du}{dx} + V\frac{du}{dy} &= 0 \\ \frac{1}{\rho}\frac{dp}{dy} + By + u(\Omega + \zeta) + U\frac{dv}{dx} + V\frac{dv}{dy} &= 0 \\ \frac{1}{\rho}\frac{dp}{dz} + Cz \qquad\qquad\qquad\qquad &= 0 \end{aligned}\right\} \quad \text{......(32)}.$$

Also

$$U = u + (\Omega + \zeta)y = \frac{2a^2\Omega y}{a^2 + b^2},$$

$$V = v - (\Omega + \zeta)x = -\frac{2b^2\Omega x}{a^2 + b^2}.$$

Substituting these values of u, v, U, V in (32) we obtain

$$\frac{1}{\rho}\frac{dp}{dx} + \left[A - 2\Omega(\Omega + \zeta)\frac{a^2 - b^2}{a^2 + b^2} + \Omega^2\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2 - \zeta^2\right]x = 0,$$

$$\frac{1}{\rho}\frac{dp}{dy} + \left[B + 2\Omega(\Omega + \zeta)\frac{a^2 - b^2}{a^2 + b^2} + \Omega^2\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2 - \zeta^2\right]y = 0,$$

$$\frac{1}{\rho}\frac{dp}{dz} + Cz \qquad = 0;$$

the integral of which is

$$\frac{p}{\rho} + \tfrac{1}{2}(Ax^2 + By^2 + Cz^2) - \Omega(\Omega + \zeta)(x^2 - y^2)\frac{a^2 - b^2}{a^2 + b^2}$$
$$+ \tfrac{1}{2}\left\{\Omega^2\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2 - \zeta^2\right\}(x^2 + y^2) = \text{const.},$$

which determines the surfaces of equal pressure.

[1] Greenhill, *Proc. Camb. Phil. Soc.* vol. III. p. 233 and vol. IV. pp. 4 and 208.

The condition that the free surface should be the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, is

$$\left.\begin{aligned} & a^2 \left\{A - 2\Omega(\Omega + \zeta)\frac{a^2 - b^2}{a^2 + b^2} + \Omega^2\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2 - \zeta^2\right\} \\ & = b^2 \left\{B + 2\Omega(\Omega + \zeta)\frac{a^2 - b^2}{a^2 + b^2} + \Omega^2\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2 - \zeta^2\right\} \\ & = Cc^2 \end{aligned}\right\} \quad \ldots(33).$$

These equations show that Aa^2 is greater than Bb^2 or Cc^2, and hence a must be the greatest axis; and therefore the greatest axis can never be the axis of rotation.

The axis of rotation will be the mean or least axis according as

$$Cc^2 > \text{or} < Bb^2,$$

that is, according as

$$\zeta^2 - 2\Omega(\Omega + \zeta)\frac{a^2 - b^2}{a^2 + b^2} - \Omega^2\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2$$

is negative or positive, that is, according as

$$\frac{\zeta}{\Omega} < \text{or} > \frac{a^2 - b^2}{a^2 + b^2}\left\{1 \pm \frac{2a}{\sqrt{(a^2 - b^2)}}\right\}.$$

If the ratio ζ/Ω is such that this inequality becomes an equality, we must have $b = c$, and the free surface will be an ovary ellipsoid rotating about an equatorial axis. This is the only case in which the free surface can be an ovary ellipsoid.

356. We must now consider the first four cases in detail.

Case (i). *Maclaurin's Ellipsoid.*

Here $a = b$, $\Omega = 0$, and (33) becomes

$$\zeta^2 a^2 = Aa^2 - Cc^2 \quad \ldots\ldots\ldots\ldots(34).$$

The free surface is therefore a planetary ellipsoid, and the liquid rotates with angular velocity ζ about the polar axis.

Now

$$A = 2\pi\rho a^2 c \int_0^\infty \frac{d\lambda}{(a + \lambda)^2 (c^2 + \lambda)^{\frac{1}{2}}},$$

$$C = 2\pi\rho a^2 c \int_0^\infty \frac{d\lambda}{(a^2 + \lambda)(c^2 + \lambda)^{\frac{3}{2}}}.$$

Putting $\nu = (1 - e^2)^{\frac{1}{2}}/e$ in (14) and (15) of § 148, we obtain

$$A = 2\pi\rho e^{-3}(1 - e^2)^{\frac{1}{2}}\{\sin^{-1} e - e(1 - e^2)^{\frac{1}{2}}\},$$

$$C = 4\pi\rho e^{-3}\{e - (1 - e^2)^{\frac{1}{2}}\sin^{-1} e\},$$

where e is the excentricity, whence (34) becomes

$$\zeta^2/2\pi\rho = e^{-3}(1-e^2)^{\frac{1}{2}}\{(3-2e^2)\sin^{-1}e - 3e(1-e^2)^{\frac{1}{2}}\}\ldots(35).$$

The right-hand side of this equation can easily be seen to vanish when $e=0$ and $e=1$, and to be positive for all values of e between 0 and 1. Hence as e increases from 0 to 1, the right-hand side increases from zero to a certain maximum value, and then decreases to zero. It therefore follows that for all values of $\zeta^2/2\pi\rho$ which are less than this maximum value, there will be two ellipsoidal forms of the free surface, the excentricities of whose meridian curves are determined by the two roots of (35); when $\zeta^2/2\pi\rho$ is equal to this maximum value, there is only one ellipsoidal form; and when $\zeta^2/2\pi\rho$ is greater than this maximum value, the ellipsoidal form is impossible.

The excentricity of the ellipsoid corresponding to the maximum value of $\zeta^2/2\pi\rho$ is determined by the equation

$$(9-8e^2)\sin^{-1}e = e(9-2e^2)(1-e^2)^{\frac{1}{2}}.$$

In this put $e^2=\lambda^2/(1+\lambda^2)$ and we obtain

$$\frac{\lambda(9+7\lambda^2)}{(1+\lambda^2)(9+\lambda^2)} - \tan^{-1}\lambda = 0.$$

In order to find the root of this equation[1], denote the left-hand side by $f(\lambda)$. Let $\lambda = 2{\cdot}5$, then by the aid of the formula

$$\tan^{-1}2{\cdot}5 = \tan^{-1}2 + \tan^{-1}\tfrac{1}{12},$$

we obtain $$f(2{\cdot}5) = {\cdot}0025.$$

Let $$\lambda = 2{\cdot}5 + y,$$

then approximately

$$y = -f(2{\cdot}5)/f'(2{\cdot}5),$$

also $$f'(2{\cdot}5) = -{\cdot}085 \text{ nearly};$$

therefore $$y = {\cdot}0293,$$

$$\lambda = 2{\cdot}5293.$$

Substituting this value of λ in (35), we shall obtain

$$\zeta^2/4\pi\rho = {\cdot}1123,$$

which determines the maximum value of the angular velocity.

The value of the excentricity will be found to be approximately equal to ·93.

[1] Besant's *Hydromechanics*, ch. VIII.; see also Thomson and Tait's *Nat. Phil.* vol. I. part II. p. 327, where a table is given.

357. Case (ii). *Jacobi's Ellipsoid.*

In (33) put $\Omega = 0$, and we obtain

$$(A - \zeta^2)\, a^2 = (B - \zeta^2)\, b^2 = Cc^2,$$

or
$$\zeta^2 = (Aa^2 - Cc^2)/a^2 = (Bb^2 - Cc^2)/b^2 \ldots\ldots\ldots\ldots(36).$$

In order that the value of ζ may be real it is necessary that $Aa^2 > Bb^2 > Cc^2$; hence $a > b > c$, and therefore the axis of rotation must be the *least* axis. The free surface is therefore an ellipsoid about whose least axis the liquid rotates as a rigid body; also since the volume of the ellipsoid is constant, it follows from (36) that when ζ is given there is only one ellipsoid satisfying the conditions of the problem.

From (36) we have

$$a^2b^2(A - B) + (a^2 - b^2)\, Cc^2 = 0,$$

or
$$a^2b^2 \int_0^\infty \frac{d\lambda}{(a^2+\lambda)(b^2+\lambda)P} = c^2 \int_0^\infty \frac{d\lambda}{(c^2+\lambda)P} \ldots\ldots(37),$$

or
$$\int_0^\infty \{(a^2b^2 - a^2c^2 - b^2c^2)\lambda - c^2\lambda^2\}\, P^{-3}\, d\lambda = 0.$$

If $c = 0$ the last integral is positive, and if $c = ab/(a^2+b^2)^{\frac{1}{2}}$ the integral is negative; hence c must have some value lying between 0 and $ab/(a^2+b^2)^{\frac{1}{2}}$.

According to Ivory[1], the axes must be proportional to

$$c, \quad c\sqrt{(1+\lambda^2)}, \quad c\sqrt{(1+n^2/\lambda^2)}$$

where n is a numerical quantity lying between 1 and 1·9414.

When $a = b$, Jacobi's ellipsoid coalesces with the most oblate of the two Maclaurin's ellipsoids. In order to find the excentricity of this ellipsoid, put

$(a^2+\lambda)^{\frac{1}{2}} = aev$ in (37) and integrate, and we shall obtain

$$(\tfrac{3}{4} + 2e^2 - 2e^4)\sin^{-1} e = e(1-e^2)^{\frac{1}{2}}(\tfrac{3}{4} + \tfrac{5}{2}e^2).$$

By trial and error it can be shown that this equation has one real root lying between 0 and 1, which is approximately equal to ·8127, and the corresponding value of $\zeta^2/4\pi\rho$ is ·0934. Hence when $\zeta^2/4\pi\rho$ lies between 0 and ·0934, there are three possible forms of the free surface, viz., the two ellipsoids of revolution and an ellipsoid with three unequal axes; when $\zeta^2/4\pi\rho$ lies between ·0934 and ·1123 the two revolutional ellipsoids are the only ellipsoidal forms possible.

[1] *Phil. Trans.* 1838.

358. Case (iii). *Dedekind's Ellipsoid.*

In (33) put $\Omega + \dot{\zeta} = 0$, and we obtain

$$a^2\left(A - \frac{4a^2b^2\zeta^2}{a^2+b^2}\right) = b^2\left(B - \frac{4a^2b^2\zeta^2}{a^2+b^2}\right) = Cc^2.$$

Hence the ellipsoidal boundary is stationary, but there is an internal motion of the particles, which from (31) is determined by the equations

$$\dot{x} = \frac{2a^2\Omega y}{a^2+b^2},$$

$$\dot{y} = -\frac{2b^2\Omega x}{a^2+b^2}.$$

Whence $$\ddot{x} + \frac{4a^2b^2\Omega^2 x}{(a^2+b^2)^2} = 0,$$

therefore $$x = A\cos(kt+\alpha),$$

and $$y = -Aa^{-1}b\sin(kt+\alpha),$$

where $$k = 2ab\Omega/(a^2+b^2).$$

Hence if x_0, y_0, z_0 are the initial co-ordinates of the element of liquid whose co-ordinates at time t are x, y, z, we obtain,

$$x = x_0\cos kt + ab^{-1}y_0\sin kt,$$

$$y = -a^{-1}bx_0\sin kt + y_0\cos kt,$$

$$z = z_0.$$

In Dedekind's ellipsoid the quantity $2ab\zeta/(a^2+b^2)^{\frac{1}{2}}$ takes the place of ζ in Jacobi's ellipsoid, and it can be shown in the same manner that we must have $0 < c < ab/(a^2+b^2)^{\frac{1}{2}}$ and that there is only one ellipsoid satisfying the conditions. When $a = b$ Dedekind's ellipsoid coalesces with the limiting Jacobian ellipsoid, and therefore when $\zeta^2/4\pi\rho > \cdot 0934$ Dedekind's ellipsoid is impossible.

359. Case (iv). *The Irrotational Ellipsoid.*

In (33) put $\zeta = 0$, and we obtain

$$\frac{\Omega^2(a^2+b^2)^2}{a^2-b^2} = \frac{Aa^2 - Cc^2}{a^2(a^2+3b^2)} = \frac{Cc^2 - Bb^2}{b^2(3a^2+b^2)}.$$

The motion of the liquid is therefore irrotational, and is the same as might be generated from rest by filling an ellipsoidal cavity with liquid, and setting it in rotation about the axis c. Moreover, in order that Ω may be real, we must have $Cc^2 > Bb^2$, hence $c > b$, and the axis of rotation must be the mean axis. In this case the revolutional form is evidently impossible.

360. Case (vi). *Riemann's Ellipsoid.*

In order to investigate the most general kind of steady motion of which a liquid ellipsoid is capable, we must employ the general equations of motion. Putting $d(\xi a^{-1})/dt$, &c. equal to zero, we obtain from (5)

$$\frac{\Omega_1}{\xi(b^2+c^2)} = \frac{\Omega_2}{\eta(c^2+a^2)} = \frac{\Omega_3}{\zeta(a^2+b^2)} = \text{const.} = \frac{1}{\mu} \quad \ldots\ldots(38).$$

Also $\quad \omega_1 = \Omega_1 + \xi = \Omega_1 \{\mu/(b^2+c^2)+1\}$, &c. &c.

From (7) we have

$$h_1/\omega_1 = h_2/\omega_2 = h_3/\omega_3.$$

Substituting the values of h_1, h_2, h_3 in terms of ω_1, Ω_1 &c., from (6) it will be found that (38) are equivalent to the following three equations:

$$\mu^2 - (2a^2 - b^2 - c^2)\mu + (c^2+a^2)(a^2+b^2) - 4a^4 = 0$$
$$\mu^2 - (2b^2 - c^2 - a^2)\mu + (a^2+b^2)(b^2+c^2) - 4b^4 = 0$$
$$\mu^2 - (2c^2 - a^2 - b^2)\mu + (b^2+c^2)(c^2+a^2) - 4c^4 = 0.$$

These three equations cannot co-exist, hence one of the three pairs of quantities Ω_1, ξ &c. must be zero. *Hence steady motion is impossible unless the instantaneous axis of rotation lies in a principal plane.*

361. Let us therefore suppose that $\Omega_1 = \xi = 0$. From the fourth and fifth of (18) we obtain,

$$\frac{v'^2}{v^2} = \frac{(2a-b-c)(2a+b-c)}{(2a+b+c)(2a-b+c)},$$

$$\frac{w'^2}{w^2} = \frac{(2a-b-c)(2a-b+c)}{(2a+b+c)(2a+b-c)}.$$

Let

$$\left.\begin{aligned} \frac{v^2}{(2a+b+c)(2a-b+c)} &= \frac{v'^2}{(2a-b-c)(2a+b-c)} = S \\ \frac{w^2}{(2a+b+c)(2a+b-c)} &= \frac{w'^2}{(2a-b-c)(2a-b+c)} = T \end{aligned}\right\} \ldots(39).$$

Substituting in the first three of (18) we obtain

$$(4a^2 - b^2 - 3c^2)S + (4a^2 - 3b^2 - c^2)T = \tfrac{1}{4}A - \tfrac{1}{2}\sigma/a^2 \ldots(40).$$

$$\left.\begin{aligned} (b^2-c^2)T &= \tfrac{1}{4}B - \tfrac{1}{2}\sigma/b^2 \\ (c^2-b^2)S &= \tfrac{1}{4}C - \tfrac{1}{2}\sigma/c^2 \end{aligned}\right\} \ldots(41).$$

Solving these equations we obtain

$$\sigma = \frac{\pi\rho a^2b^2c^2}{D}\int_0^\infty \left\{\frac{2\lambda + 4a^2 - b^2 - c^2}{(b^2+\lambda)(c^2+\lambda)} + \frac{1}{a^2+\lambda}\right\}\frac{d\lambda}{P'} \quad\ldots\ldots\ldots\ldots(42).$$

$$S = \frac{\pi\rho\,(b^2-a^2)}{2D\,(b^2-c^2)}\int_0^\infty \left\{\frac{4a^2 - c^2 + b^2}{c^2+\lambda} - \frac{b^2}{a^2+\lambda}\right\}\frac{\lambda d\lambda}{(b^2+\lambda)P'} \quad\ldots\ldots(43).$$

$$T = \frac{\pi\rho\,(c^2-a^2)}{2D\,(c^2-b^2)}\int_0^\infty \left\{\frac{4a^2 - b^2 + c^2}{b^2+\lambda} - \frac{c^2}{a^2+\lambda}\right\}\frac{\lambda d\lambda}{(c^2+\lambda)P'} \quad\ldots\ldots\ldots(44),$$

where $$D = 4a^4 - a^2(b^2+c^2) + b^2c^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(45).$$

$$P' = P/abc.$$

362. We must now find the relations between a, b, c in order that these equations may give real values of v, v', w, w' and also make σ positive.

In order that $(v'/v)^2$ and $(w'/w)^2$ should be positive, it is necessary and sufficient that

$$a > \tfrac{1}{2}(b+c) \text{ or } < \tfrac{1}{2}(b-c),$$

and there are three cases to be considered.

Case I. $$a > \tfrac{1}{2}(b+c).$$

In this case it is easily seen that D and both the integrals on the right-hand sides of (43) and (44) are positive, for

$$D = a^2\{4a^2 - (b+c)^2\} + bc\,(2a^2+bc),$$

also the integral (43)

$$= \int_0^\infty \{(4a^2-c^2)\lambda + a^2(4a^2+b^2-c^2) - b^2c^2\}\frac{\lambda d\lambda}{a^2b^2c^2P'^3}.$$

Since $2a > b+c$, then $4a^2 > c^2$; also

$$4a^2 + b^2 - c^2 > (b+c)^2 + b^2 - c^2 > 2b\,(b+c),$$

therefore $a^2(4a^2+b^2-c^2) > 2a^2b\,(b+c) > \tfrac{1}{2}b\,(b+c)^3 > b^2c^2$.

Hence the above integral is positive; similarly by interchanging b and c, it is seen that the integral on the right-hand side of (44) is also positive. If now a increase from $\tfrac{1}{2}(b+c)$ to ∞, T will be always positive provided $b > c$, but S will be positive only so long as $a < b$; hence in this case we must have

$$\left.\begin{array}{c} b > a > \tfrac{1}{2}(b+c) \\ b > c \end{array}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46).$$

b must therefore be the greatest axis, but a may be either the mean or the least axis.

Case II. $\qquad a < \frac{1}{2}(b-c),\ \ a > c.$

Since $2a < b - c$, S must be negative and T positive; now

$$(4a^2 - b^2)\lambda + a^2(4a^2 + c^2 - b^2) - b^2c^2$$

is always negative, and therefore T can never be positive unless $D(c^2 - a^2)$ is positive, which requires that D should be negative and therefore

$$c^2 < a^2(b^2 - 4a^2)/(b^2 - a^2),$$

which is always possible since the right-hand side of this inequality $< a^2$. Also since $c^2 + \lambda < a^2 + \lambda$ and $4a^2 > c^2$, the integral on the right-hand side of (43) will always be positive and therefore S will be negative.

This case may be further divided into two sub-cases.

(i). The first condition may be written $c < b - 2a$, which requires that $b > 2a$, whence b must be the greatest axis. Now if $b > a(\sqrt{3}+1)$, it can easily be shown that

$$(b - 2a)^2 > a^2(b^2 - 4a^2)/(b^2 - a^2),$$

hence the conditions may be written

$$\left.\begin{array}{l} b > a(\sqrt{3}+1) \\ c < a\sqrt{(b^2 - 4a^2)/(b^2 - a^2)} \end{array}\right\} \qquad\qquad (47).$$

(ii). But if $a(\sqrt{3}+1) > b > 2a$, then

$$(b - 2a)^2 < a^2(b^2 - 4a^2)/(b^2 - a^2),$$

and the conditions become

$$\left.\begin{array}{r} a(\sqrt{3}+1) > b > 2a \\ c < b - 2a \end{array}\right\} \qquad\qquad (48).$$

Case III. $\qquad a < \frac{1}{2}(b-c),\ \ a < c.$

The second condition requires that

$$c^2 > a^2(b^2 - 4a^2)/(b^2 - a^2),$$

and therefore D and T are both positive. The value of S remains negative so long as $a < \frac{1}{2}c$, and becomes positive when $a = c$, and therefore the integral becomes positive for some value of a which lies between $\frac{1}{2}c$ and c. Hence the conditions in this case reduce to

$$a < \tfrac{1}{2}(b-c),\ \ a < k,$$

where

$$\tfrac{1}{2}c < k < c.$$

Lastly, in order that the motion may be possible without the

aid of an external pressure, it is necessary that σ should be always positive. The value of σ may be expressed in the form

$$\sigma = \pi\rho D^{-1}\int_0^\infty (3\lambda^2 + 6\lambda a^2 + D)\, P'^{-3}\, d\lambda.$$

In the first and third cases D and therefore σ is always positive, but in the second case where D is negative a further limitation is required.

On the Stability of an Ellipsoid.

363. The question of the stability of a liquid ellipsoid has been discussed by Sir W. Thomson[1], and a very elaborate investigation of this question has been made by Poincaré[2], to which the reader must be referred for complete information on the subject. The problem in its most general form is this. A mass of liquid is rotating about its centre of inertia in a state of steady motion, under the influence of its own attraction, in such a manner that the form of the free surface is an ellipsoid, and a disturbance of any kind is communicated to the liquid; it is required to determine whether the resulting motion is stable or unstable.

In the present section, we shall not attempt to deal with the problem in its most general form, but the investigation will be confined to the consideration of the stability of a liquid ellipsoid which in steady motion is rotating about a principal axis, and which is subjected to a disturbance such that the free surface in the beginning of the disturbed motion is an ellipsoid[3]. A disturbance of this character may be communicated by enclosing the liquid ellipsoid in a case which is subjected to an impulsive couple about any diameter together with a deformation of its surface, and is therefore equivalent to a disturbance produced by an impulsive pressure communicated to the free surface of the liquid.

[1] Thomson and Tait, vol. I. part II. pp. 329 and 333; *Proc. Roy. Soc. Edin.* vol. XI. p. 610.

[2] *Acta Mathematica*, vol. VII. p. 259.

[3] Riemann, *Gatt. Abhand.* vol. IX.; see also *Proc. Lond. Math. Soc.* vol. XIX. p. 46. The investigation given in the latter paper respecting the stability of Maclaurin's ellipsoid is erroneous.

364. By (23), the potential energy of an ellipsoidal mass of gravitating liquid of mass M and uniform of density ρ is

$$U = D - \tfrac{2}{5} M\pi\rho abc \int_0^\infty \frac{d\lambda}{P},$$

where $P = \sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}$, and D is a constant. Let R be the radius of a sphere of equal volume, then

$$U = 0 \text{ when } a = b = c = R,$$

therefore
$$D = \tfrac{4}{5} M\pi\rho R^2,$$

and
$$U = \tfrac{4}{5} M\pi\rho R^2 - \tfrac{2}{5} M\pi\rho abc \int_0^\infty \frac{d\lambda}{P} \quad\text{..............(49).}$$

Now U is evidently positive; hence the integral must be a maximum when $a = b = c = R$, and will become indefinitely small when any one of the axes of the ellipsoid becomes infinitely small or infinitely large.

Let $2c$ be the axis of rotation, and let

$$E = \tfrac{1}{10} M \left\{ \frac{\omega_3^2 (a^2-b^2)^2}{a^2+b^2} + \frac{4a^2b^2\zeta^2}{a^2+b^2} - 4\pi\rho\, abc \int_0^\infty \frac{d\lambda}{P} \right\} \quad\text{......(50).}$$

By (22) and (23) E is the variable part of the energy of a mass of liquid whose free surface is constrained to maintain a fixed ellipsoidal form and which is rotating about the axis c. In steady motion ω_3 and ζ, and therefore E, are certain functions of a, b, c; let E_0 be the value of E in steady motion.

Let a disturbance (which for brevity will be called an ellipsoidal disturbance) be communicated to the liquid by means of an impulsive pressure applied to its free surface, which is such that in the beginning of the disturbed motion the free surface is a slightly different ellipsoid. Then, if $E_0 + \delta E$ is the energy of the disturbed motion, we obtain by (22) and (23),

$$\delta E = \tfrac{1}{10} M \left[\dot{a}^2 + \dot{b}^2 + \dot{c}^2 + \frac{\omega_1^2 (b^2-c^2)^2}{b^2+c^2} + \frac{\omega_2^2 (c^2-a^2)^2}{c^2+a^2} + \frac{4b^2c^2\xi^2}{b^2+c^2} + \frac{4c^2a^2\eta^2}{c^2+a^2} \right] + E - E_0.$$

All the terms in square brackets are positive, and in the beginning of the disturbed motion are small quantities; hence, if $E > E_0$, these terms must remain small quantities and the free surface can never deviate far from its form in steady motion, and the motion is therefore stable. But, if $E < E_0$, the terms in square brackets may

become a finite positive quantity, and the difference $E-E_0$ may become a finite negative quantity, such that the difference between the two sets of terms always remains equal to the infinitesimal quantity δE. When this is the case the free surface may deviate far from its form in steady motion, and the motion may be unstable.

Hence, for the particular kind of disturbance which we are considering, the condition of stability requires that the energy in steady motion should be a minimum. Or, in other words, if the steady motion is stable, it must be impossible by any kind of ellipsoidal disturbance to abstract energy from the system.

365. Let the disturbing pressure be divided into two parts p_1, p_2, the former of which produces a variation of the axes and no change in the angular momentum, whilst the latter produces no instantaneous variation of the axes but changes the angular momentum. The resultant of p_2 will consist of a couple G, and a single force, which produces a translation of the whole mass of liquid, and which it is unnecessary to consider. If the axis of this couple lie in the principal plane, which is perpendicular to the axis of rotation in steady motion, the energy will be evidently increased by its application; but, if the axis of the couple does not lie in this principal plane, the component of the couple about the axis of rotation may diminish the energy if it acts in the opposite direction to that of rotation, in which case the motion will be unstable.

In Maclaurin's ellipsoid the component of the couple about the axis of rotation necessarily vanishes, since p_2 always passes through the axis of rotation; the case of Dedekind's ellipsoid, in which the free surface is stationary, will be considered later on.

Hence, so far as the action of p_2 is concerned, Jacobi's ellipsoid, the irrotational ellipsoid, and the ellipsoids belonging to the general class V., including the ovary ellipsoid rotating about an equatorial axis, but excluding Dedekind's ellipsoid, are stable whenever the couple component about the axis of rotation of the disturbing pressure either vanishes or is in the same direction as the rotation; but when this is not the case the motion may be unstable.

In the case of Dedekind's ellipsoid, by (50),

$$E_0=\tfrac{2}{5}M\left\{\frac{a^2b^2\zeta^2}{a^2+b^2}-\pi\rho abc\int_0^\infty\frac{d\lambda}{P}\right\},$$

where $$\frac{4a^2b^2\zeta^2}{a^2+b^2} = \frac{Aa^2 - Cc^2}{a^2} = \frac{Bb^2 - Cc^2}{b^2},$$

and the effect of a disturbing couple about the axis of rotation will be to increase the energy by the quantity

$$\frac{M\omega_3^2 (a^2-b^2)^2}{10(a^2+b^2)},$$

whence $E > E_0$, and therefore the motion so far as this kind of disturbance is concerned is stable.

366. We must now consider the disturbance p_1 which produces a variation of the axes. From the last two of (18) we obtain

$$(a-b)^2 w = \text{const.} = \tau, \quad (a+b)^2 w' = \text{const.} = \tau' \text{......(51).}$$

whence, from (9),

$$\frac{\zeta}{c} = \frac{\tau' - \tau}{2abc}, \quad h_3 = \tfrac{1}{5}M(\tau' + \tau) \text{ ....(52).}$$

Also, from (6)

$$h_3 = \frac{M}{5(a^2+b^2)} \{(a^2-b^2)^2 \omega_3 + 4a^2b^2\zeta\},$$

whence $$E = \tfrac{1}{5}M \left\{\frac{\tau^2}{(a-b)^2} + \frac{\tau'^2}{(a+b)^2} - 2H\right\} \text{............(53),}$$

where $$H = \pi\rho abc \int_0^\infty \frac{d\lambda}{P}.$$

Also putting $\ddot{a}$, $\ddot{b}$, $\ddot{c}$ each equal to zero in the first three of (18) and taking account of (51) we obtain

$$\left.\begin{aligned} 0 &= \tfrac{1}{2}Cc - \frac{\sigma}{c} \\ \frac{\tau'^2}{(a+b)^3} + \frac{\tau^2}{(a-b)^3} &= \tfrac{1}{2}Aa - \frac{\sigma}{a} = \frac{1}{2a}(Aa^2 - Cc^2) \\ \frac{\tau'^2}{(a+b)^3} - \frac{\tau^2}{(a-b)^3} &= \tfrac{1}{2}Bb - \frac{\sigma}{b} = \frac{1}{2b}(Bb^2 - Cc^2) \end{aligned}\right\} \text{..... (54).}$$

Whence (53) becomes

$$\begin{aligned} E_0 &= \tfrac{1}{2}(Aa^2 + Bb^2 - 2Cc^2) - 2H \\ &= -H - \tfrac{3}{2}Cc^2 \text{ ....(55),} \end{aligned}$$

since $$Aa^2 + Bb^2 + Cc^2 = 2H.$$

Whence E_0 is a finite *negative* quantity.

The constants τ, τ' express the fact that the angular momentum

and the vorticity are unchanged during the motion; also since the disturbance p_1 does not change the angular momentum or vorticity, these constants must have the same values as in steady motion.

Since the volume of the ellipsoid is constant, the conditions that E may be a minimum require that

$$\left.\begin{aligned}\frac{dE}{da}-\frac{c}{a}\frac{dE}{dc}=0\\ \frac{dE}{db}-\frac{c}{b}\frac{dE}{dc}=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(56).$$

On performing the differentiations it will be found that (56) lead to (54); hence the first conditions are satisfied.

We must now enquire whether, in the general case, E has a minimum value when τ and τ' are unchanged by the disturbance.

Let $z=5E/M$, $R^3=abc$, $x=a$, $y=b$, then

$$z=\frac{\tau^2}{(x-y)^2}+\frac{\tau'^2}{(x+y)^2}-2\pi\rho R^3\int_0^\infty\frac{d\lambda}{\sqrt{(x^2+\lambda)(y^2+\lambda)(R^6/x^2y^2+\lambda)}}\ldots(57).$$

Since a, b, c, are positive, and a is never less than b, we have to examine the form of the surface (57) between the planes $y=0$, $x-y=0$.

First suppose τ is not zero.

When $x=y$, $z=\infty$. If y has any finite value $<$ or $=x$, then, as x increases from y to infinity, z diminishes, and the value of E_0 in steady motion shows that z will vanish and become negative, and when x is very large z is very small. Moreover, z can never become equal to $-\infty$ for any values of x or y, and when x and y are *both* very large z is very small, unless $x-y$ is small.

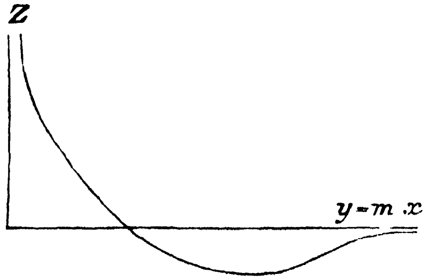


Fig. 1.

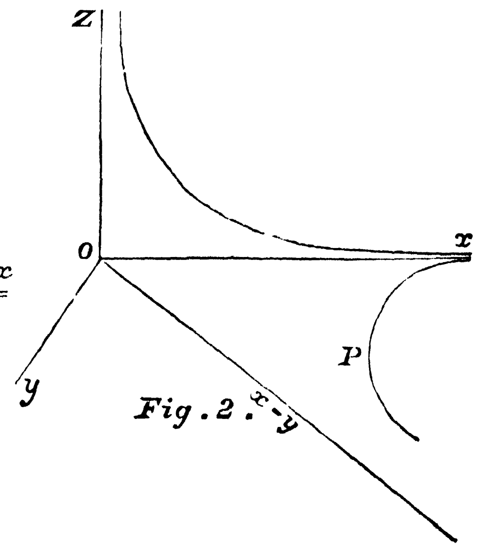


Fig. 2.

A general idea of the form of the surface may be obtained from the accompanying figures. Fig. 1 is the curve of section made by the plane $y = mx$, $m < 1$; and Fig. 2 shows the curves of section made by the planes xz and xy. The surface cuts the place of xy along the curve xP, and the sheet underneath this plane gradually bends upwards towards the plane.

It therefore follows that in this case z must have a minimum value, which is given by (55).

367. If $\tau = 0$, it follows from (54) either that $a = b$, which is the case of Maclaurin's ellipsoid, or the axes of the ellipsoid must be connected by the equation $(Aa^2 - Cc^2)/a = (Bb^2 - Cc^2)/b$.

We shall now show that Maclaurin's ellipsoid is unstable if the excentricity exceeds a certain value.

In steady motion

$$\tau = 0, \quad \tau'/(a+b)^2 = w' = \tfrac{1}{2}\zeta.$$

Let $Q = Aa^2 - Cc^2$, $R = Bb^2 - Cc^2$, then omitting the factor $2\pi\rho abc$ in A, B and C, the condition that Maclaurin's ellipsoid should be stable for an ellipsoidal disturbance, is that E should be a minimum in steady motion where

$$E = \frac{\tau'^2}{(a+b)^2} - 2H.$$

Putting $E_a = dE/da$ &c. we obtain

$$E_a = \frac{Q}{a} - \frac{2\tau'^2}{(a+b)^3}, \quad E_b = \frac{R}{b} - \frac{2\tau'^2}{(a+b)^2},$$

$$E_{aa} = \frac{1}{a}\left(\frac{dQ}{da} - \frac{c}{a}\frac{dQ}{dc}\right) - \frac{Q}{a^2} + \frac{6\tau'^2}{(a+b)^4},$$

$$E_{ab} = \frac{1}{a}\left(\frac{dQ}{db} - \frac{c}{b}\frac{dQ}{dc}\right) + \frac{6\tau'^2}{(a+b)^4},$$

where b is to be put equal to a after differentiation. Now when $a = b$, $Q = R$ and $E_{aa} = E_{bb}$, therefore

$$\delta E = \tfrac{1}{2}(E_{aa}\delta a^2 + 2E_{ab}\delta a\delta b + E_{aa}\delta b^2)$$
$$= \tfrac{1}{4}(E_{aa} + E_{ab})(\delta a + \delta b)^2 + \tfrac{1}{4}(E_{aa} - E_{ab})(\delta a - \delta b)^2.$$

Now $$a^2(E_{aa} + E_{ab}) = a\left(\frac{dQ}{da} + \frac{dQ}{db}\right) - 2c\frac{dQ}{dc} + 2Q.$$

On reduction it will be found that the right-hand side is equal to

$$9e(1-e^2)^{\frac{1}{2}}(3-2e^2)-(27-36e^2+8e^4)\sin^{-1}e,$$

which is positive for all values of e between zero and unity. But

$$a^2(E_{aa}-E_{ab})=\left(\frac{dQ}{da}-\frac{dQ}{db}\right)a-Q,$$

$$=e(1-e^2)^{\frac{1}{2}}(3+4e^2)-(3+2e^2-4e^4)\sin^{-1}e\ldots\ (58)$$

on reduction. The right-hand side is positive when $e=0$, and negative when $e=1$, whence Maclaurin's ellipsoid becomes unstable when the excentricity exceeds the root of the equation obtained by equating the right-hand side of (58) to zero.

The equation determining the excentricity of the revolutional ellipsoid which coincides with the limiting Jacobian ellipsoid has been found in § 357, and on comparing it with (58), it will be seen that the excentricity of this ellipsoid is somewhat less than the ellipsoid which is unstable.

This result was first obtained by Riemann[1].

In the last edition of Thomson and Tait's *Natural Philosophy*, vol. I. part II. p. 333, it is stated that Maclaurin's ellipsoid is stable or unstable, according as the excentricity is less or greater than the ellipsoid which coalesces with the limiting Jacobian ellipsoid; i.e. according as $e <$ or $>$ ·8127. Unfortunately no proof of this statement is given, but if it is correct, the disturbance which produces instability cannot be an ellipsoidal disturbance, but must be one of a more general character.

368. Poincaré[2] has shown that when a mass of liquid is rotating about a fixed axis as a rigid body, the problem of determining the small oscillations is reducible to the solution of a single equation.

Let the axis of rotation in steady motion be the axis of z, and let the axes of x and y be any two perpendicular axes which are rotating with angular velocity ω. Then if the disturbed motion be referred to the *same* axes, the equations of motion are

$$\frac{\partial u}{\partial t}-\omega v=\frac{dQ}{dx},\quad \frac{\partial v}{\partial t}+\omega u=\frac{dQ}{dy},\quad \frac{\partial w}{\partial t}=\frac{dQ}{dz},$$

[1] *Gott. Abhand.* vol. IX. § 9.

[2] *Acta Math.* vol. VII. p. 356.

where $$\partial/\partial t = d/dt + Ud/dx + Vd/dz + Wd/dz,$$

$$Q = -p/\rho + V';$$

U, V, W being the velocities of the liquid relative to the moving axes, and V' is the potential. Also

$$U = u + \omega y, \quad V = v - \omega x, \quad W = w.$$

Since the liquid is rotating as a rigid body in steady motion, U, V, and W are all zero, hence in the disturbed motion U, V, W are all small quantities; if therefore we put

$$\psi = Q + \tfrac{1}{2}\omega^2(x^2 + y^2),$$

the equations of disturbed motion are

$$\frac{dU}{dt} - 2\omega V = \frac{d\psi}{dx}, \quad \frac{dV}{dt} + 2\omega U = \frac{d\psi}{dy}, \quad \frac{dW}{dt} = \frac{d\psi}{dz}.$$

The equation of continuity is

$$\frac{dU}{dx} + \frac{dV}{dy} + \frac{dW}{dz} = 0,$$

whence $$\nabla^2\psi = 2\omega\left(\frac{dU}{dy} - \frac{dV}{dx}\right),$$

$$\frac{d}{dt}(\nabla^2\psi) = 2\omega\frac{d}{dy}\left(\frac{d\psi}{dx} + 2\omega V\right) - 2\omega\frac{d}{dx}\left(\frac{d\psi}{dy} - 2\omega U\right),$$

$$= 4\omega^2\left(\frac{dU}{dx} + \frac{dV}{dy}\right) = -4\omega^2\frac{dW}{dz}.$$

Hence $$\frac{d^2}{dt^2}(\nabla^2\psi) = -4\omega^2\frac{d^2\psi}{dz^2},$$

which is Poincaré's equation.

If we assume that the time enters into ψ in the form of the factor ϵ^{int}, this becomes

$$\nabla^2\psi - \frac{4\omega^2}{n^2}\frac{d^2\psi}{dz^2} = 0.$$

Putting $z = z'\sqrt{(1 - 4\omega^2/n^2)}$, this becomes

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} + \frac{d^2\psi}{dz'^2} = 0.$$

The problem is therefore reduced to finding a solution of Laplace's equation within the surface which is derived from the original surface by writing $z'\sqrt{(1 - 4\omega^2/n^2)}$ for z.

The solution of this equation subject to the boundary conditions will lead to an equation for determining n, which will show whether the motion is stable or unstable.

The oscillations of an elliptic cylinder[1] and of an elastic spherical shell containing liquid[2], have been worked out by this method by Mr Love.

EXAMPLES.

1. An infinite cylindrical mass of liquid is rotating about its axis with angular velocity $\Omega + \zeta$, under the influence of its own attraction, where ζ is the molecular rotation. Prove that a possible form of the free surface is an elliptic cylinder, and that if a and b be the semi-axes of the cross section,

$$(\Omega + \zeta)^2 + \frac{4a^2b^2\zeta^2}{(a^2+b^2)^2} = \frac{4\pi\rho ab}{(a+b)^2}.$$

2. In the last example prove that the paths of the particles of liquid relatively to the axes of the cross section are in general pericycloids, which (i) when $\zeta(a^2+b^2) = \Omega(a^2-b^2)$ are epicycloids; (ii) when $\Omega + \zeta = 0$ are ellipses; (iii) when $\Omega = 0$ or $(\Omega+\zeta)(a^2+b^2) = \pm 2ab\Omega$ are circles.

3. A spheroidal shell whose equatorial and polar axes are $2a$ and $2c$, and whose mass may be neglected, is filled with liquid and is rotating about its centre of inertia. The motion of the liquid at every instant is such that it could be instantaneously generated by means of the first two operations explained in § 342. Prove that

$$\xi^2 + \eta^2 = L - \frac{a^2\zeta^2}{c^2},$$

$$\Omega_1^2 + \Omega_2^2 = M + \frac{(a^2+c^2)^2\zeta^2}{2c^2(a^2-c^2)},$$

$$\Omega_1\xi + \Omega_2\eta = N + \frac{(a^2+c^2)\zeta^2}{4c^2},$$

where L, M, N are constants depending on the initial motion.

Prove also that ζ can be expressed in terms of the time in terms of elliptic functions, except when $LM = N^2$, or $c = 3a$, when it is expressible by means of circular functions.

[1] *Quart. Journ.* vol. XXIII. p. 158.
[2] *Proc. Lond. Math. Soc.* vol. XIX.

4. In the case of Jacobi's ellipsoid, prove that the mean pressure throughout the liquid is $\frac{2}{5}$ of the pressure at the centre of the ellipsoid; and that if the equation of the free surface is $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, and M is the mass of the liquid, the kinetic energy of the system is

$$\tfrac{1}{10}M(Aa^2 + Bb^2 - 2Cc^2).$$

5. In the case of Maclaurin's spheroid, prove that any given mass of the liquid may be annihilated without disturbing the motion of the rest, provided the annihilated mass is bounded by the external surface and either of the two other spheroids, but that a similar theorem does not hold for laws of attraction other than that of the inverse square of the distance.

6. Prove that if a rigid ellipsoidal shell be filled with two homogeneous gravitating liquids of different densities, the denser liquid will form a nucleus in the shape of an ellipsoid; and that if the shell be made to revolve with constant angular velocity about *any* given fixed axis, a possible form of the nucleus when the liquids are in relative equilibrium will be an ellipsoid not co-axial with the external surface.

7. A rigid shell in the form of an ellipsoid of revolution is filled with two homogeneous gravitating liquids of different densities which do not mix, and the whole system is rotating uniformly in relative equilibrium round the axis of the shell. Prove that a possible form of the surface of separation is a spheroid, and find the equation connecting the excentricity with the angular velocity.

8. A mass of attracting liquid which is at rest, is enclosed in an ellipsoidal case. Prove that if the case be removed the liquid will move so as always to preserve the ellipsoidal form.

In the case of a spheroid, prove that if a be the axis of figure,

$$\dot{a}^2 = \frac{8\pi r^3 a^3}{r^3 + 2a^3}(\Omega - \Omega_0),$$

where $\Omega = \int_0^\infty \frac{d\lambda}{(c^2+\lambda)(a^2+\lambda)^{\frac{1}{2}}}$, Ω_0 is the value of Ω in one position of rest, and r is the radius of the sphere whose volume is equal to that of the liquid.

Hence show that if the two extreme values of a be $r \operatorname{cosec}^{\frac{2}{3}}\phi$, and $r \sin^{\frac{2}{3}}\theta$, the relation between θ and ϕ will be

$$\left(\frac{\pi}{2} - \theta\right)\frac{\sin^{\frac{1}{3}}\theta}{\cos\theta} = \frac{\sin^{\frac{1}{3}}\phi}{\cos\phi}\log\cot\tfrac{1}{2}\phi.$$

9. In Maclaurin's spheroid, find the ellipticity ϵ in terms of the density ρ and the angular velocity ω when the free surface is nearly spherical; and show that the whole pressure on an equatorial plane is approximately equal to $(5-6\epsilon)\,\pi^2\rho^2a^4/15$ astronomical units of force, where a is the equatorial radius.

10. In Jacobi's ellipsoid prove that gravity on the surface is inversely proportional to the perpendicular on to the tangent plane, and that the total stress across any central section is proportional to the area of the section.

11. If two concentric approximately spherical masses of fluid of densities (astronomical) ρ and $\rho+\rho'$, the denser being inside, be rotating round an axis with angular velocity n, and if a, a' be the mean radii of the outer and inner surfaces, and if the equations of the surfaces be $r=a(1+\sigma)$, $r'=a'(1+\sigma')$, prove that σ, σ' are given by the equations

$$(\tfrac{2}{5}\rho'+\rho)\,\sigma'-\tfrac{3}{5}\rho\sigma=\{\tfrac{2}{5}\rho+\rho'(a'/a)^3\}\,\sigma-\tfrac{3}{5}\rho'(a'/a)^5\sigma'=\tfrac{3}{8}n^2\pi^{-2}(\tfrac{1}{3}-\cos^2\theta).$$

12. Prove that if a thin case in the form of an ellipsoid of revolution be filled with liquid which is rotating as if rigid about its axis, the motion is unstable, if the length of the polar axis lies between one and three times the length of the equatorial axis.

13. A quantity of liquid of density ρ is enclosed in a case, which may be either an oblate or prolate spheroid, and is rotating about its polar axis like a rigid body with angular velocity ζ. Prove that if the case be removed, it will be impossible for the free surface to retain the spheroidal form unless initially $\zeta^2/2\pi\rho<1$. Prove also that if $2c$ be the length of the polar axis, the free surface will cease to be spheroidal, if at any period of the subsequent motion

$$\frac{\zeta^2}{2\pi\rho}>1+\frac{3\dot{c}^2}{8\pi\rho c^2}.$$

14. A liquid spheroid of small ellipticity ϵ is rotating about its axis like a rigid body; prove that the angular velocity is equal to $4\,(\pi\epsilon/15)^{\frac{1}{2}}$.

15. Assuming that Saturn is a spheroid of small ellipticity ϵ, and that it was originally liquid, investigate the equation

$$\tfrac{4}{5}\epsilon-\frac{3mk^3}{2Mc\,(c^2-b^2)}=\frac{\omega k^4}{M},$$

for determining the ellipticity, due partly to its own rotation ω, and partly to the disturbance caused by its ring, which is supposed to be a flat concentric circular disc, of uniform thickness and density and lying in the plane of the equator: where M is the mass of Saturn, k its mean radius; m the mass of the ring, $c \pm b$ its bounding radii, and c is large compared with k.

Prove also that the value of gravity at co-latitude θ, is to equatorial gravity in the ratio

$$1 + \epsilon \cos^2\theta \; : \; 1.$$

16. Prove that Dedekind's ellipsoid may be derived from Jacobi's ellipsoid by supposing the liquid enclosed in a case, and then imparting to the case an equal and opposite angular velocity; and show that the impulsive couple which must be applied to the case, is equal to

$$\tfrac{1}{5}M\zeta\,(a^2 - b^2)^2/(a^2 + b^2).$$

17. In the irrotational ellipsoid, prove that if the liquid be suddenly solidified, the loss of energy is equal to

$$\tfrac{2}{5}M\Omega^2a^2c^2\,(a^2 - c^2)^2/(a^2 + c^2)^3,$$

where Ω is the angular velocity of the free surface before solidification.

18. Obtain the equations for determining the small oscillations of the ellipsoids included in case v, when the position of the axis of rotation is unaffected by the disturbance which is supposed to be ellipsoidal; and prove that in the case of Maclaurin's ellipsoid, the period T of oscillation is determined by the equation

$$\{4\pi^2/T^2 - E_{aa} + E_{ab}\}\,\{(1 + 2c^2/a^2)\,4\pi^2/T^2 - E_{aa} - E_{ab}\} = 0,$$

where E is the variable part of the whole energy, and

$$E_{aa} = d^2E/da^2, \text{ \&c.}$$

CHAPTER XVI.

ON THE STEADY MOTION OF TWO MASSES OF ROTATING LIQUID.

369. WHEN a mass of liquid is rotating as a rigid body about a fixed axis under the influence of its own attraction, the condition that the motion should be steady and that the free surface should preserve an invariable form, is obtained directly from the consideration that the reversed effective forces together with the forces arising from the mutual attractions of the different portions of liquid, must form a system in statical equilibrium.

Let the axis of rotation be the axis of z; V, p, ω, ρ the attraction[1] potential, pressure, angular velocity and the density of the liquid. The equation for determining the hydrostatic pressure p gives

$$dp = \rho \left\{ \left(\frac{dV}{dx} + \omega^2 x \right) dx + \left(\frac{dV}{dy} + \omega^2 y \right) dy + \frac{dV}{dz} dz \right\};$$

whence if ρ be constant, we obtain

$$p/\rho + \text{const.} = V + \tfrac{1}{2}\omega^2 (x^2 + y^2).$$

At the free surface $p = 0$, whence the equation of the free surface is

$$V + \tfrac{1}{2}\omega^2 (x^2 + y^2) = \text{const.}\dots\dots\dots\dots\dots\dots\dots(1).$$

370. The value of V cannot be determined without knowing the form of the free surface. If any particular form of the free surface be assumed, and the resulting value of V is substituted in (1), it usually happens that it is impossible to satisfy (1); hence the problem in its most general form is one which cannot be solved

[1] Since V is the *attraction* potential, dV/dx = force in the direction of x.

by any direct method. It is however sometimes possible to obtain an approximate solution, in which the free surface differs slightly from some surface whose form is known; and we shall therefore proceed to investigate the steady motion of two approximately spherical masses of liquid which revolve like rigid bodies about a fixed axis.

The present investigation is taken from a paper by Prof. G. H. Darwin[1].

371. We must first find the potential of a homogeneous mass of gravitating matter of unit density whose free surface is approximately spherical.

Let the equation of the bounding surface be

$$r = a\,(1 + \Sigma\alpha_n Y_n)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where Y_n is a spherical surface harmonic of degree n, and α_n is a small quantity whose squares and products may be neglected.

If V, V' be the potentials at an external and internal point respectively, we may assume

$$V = \tfrac{4}{3}\pi a^3/r + \Sigma_1^\infty A_n\,(a/r)^{n+1}\,Y_n\ldots\ldots\ldots\ldots\ldots(3),$$

$$V' = -\tfrac{2}{3}\pi r^2 + \Sigma_1^\infty A_n'\,(r/a)^n\,Y_n\ldots\ldots\ldots\ldots\ldots(4),$$

for these values evidently satisfy the equations $\nabla^2 V = 0$ and $\nabla^2 V' + 4\pi = 0$ respectively. The conditions to be satisfied at the surface of the solid are

$$V = V' + \text{const.}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

$$dV/dr = dV'/dr\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Since the A's and A''s are small quantities of the order α, we may in the small terms put $r = a$, but in the first term we must give to r its full value from (2).

Substituting in (6) we obtain

$$\tfrac{8}{3}\pi a^2\,\Sigma\alpha_n Y_n - \Sigma\,(n+1)\,A_n Y_n = -\tfrac{4}{3}\pi a^2\,\Sigma\alpha_n Y_n + \Sigma_n A_n Y_n,$$

whence equating coefficients of Y_n we obtain

$$4\pi a^2\alpha_n = (n+1)\,A_n + nA_n'.$$

Similarly from (5) we obtain,

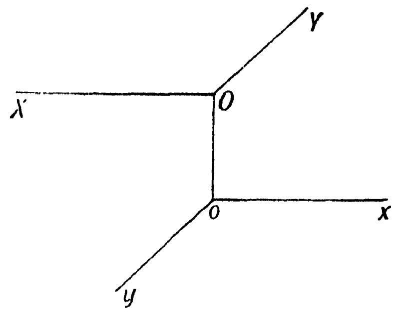
$$A_n = A_n'.$$

[1] "On the Figures of Equilibrium of Rotating Masses of Fluid," *Phil. Trans.* 1887, p. 397.

Whence $$V = \frac{4\pi a^3}{3r} + 4\pi a^2 \Sigma_1^\infty \frac{\alpha_n Y_n}{2n+1}\left(\frac{a}{r}\right)^{n+1} \quad\text{..............(7)},$$

$$V' = -\tfrac{2}{3}\pi r^2 + 4\pi a^2 \Sigma_1^\infty \frac{\alpha_n Y_n}{2n+1}\left(\frac{r}{a}\right)^{n} \quad\text{..............(8)}.$$

372. Let us now suppose that there are two masses of liquid whose free surfaces are approximately spheres, and whose centres are o and O.



Let there be two sets of rectangular axes whose origins are o and O respectively, and let the axis of z be measured from o to O, and that of Z from O to o. Let $Oo = c$; a, A the radii of the spheres whose centres are o and O respectively. Let P_n^m be an associated function whose origin is O and whose axis is OZ, and let p_n^m be a similar function when the origin is at o and the axis is oz. Let the axis of rotation be a line parallel to ox drawn through some point on Oo whose distances from O and o are D and d respectively. If Ω be the potential of the centrifugal forces, we have

$$\Omega = \tfrac{1}{2}\omega^2 (y^2 + z^2 + d^2 - 2dz).$$

Now if r, $\cos^{-1}\mu$, ϕ be polar coordinates referred to o as origin,

$$p_1(\mu) = \mu,\ p_2(\mu) = \tfrac{1}{2}(3\mu^2 - 1),\ p_2^{\,2}(\mu) = 3(1-\mu^2),$$

whence

$$\Omega = \tfrac{1}{2}\omega^2 (d^2 - 2drp_1 + \tfrac{1}{3}r^2 p_2 + \tfrac{2}{3}r^2 - \tfrac{1}{6}r^2 p_2^{\,2}\cos 2\phi).$$

Let us now put $w_n = r^n p_n$, ${}_2w_n = r^n p_n^{\,2}\cos 2\phi$, so that w_n, ${}_2w_n$ are solid harmonics of positive degree, and we obtain

$$\Omega = \tfrac{1}{2}\omega^2 (d^2 - 2dw_1 + \tfrac{1}{3}w_2 + \tfrac{2}{3}r^2 - \tfrac{1}{6}\,{}_2w_2) \quad\text{...........(9)}.$$

Similarly the value of Ω referred to the other origin O is

$$\Omega = \tfrac{1}{2}\omega^2 (D^2 - 2DW_1 + \tfrac{1}{3}W_2 + \tfrac{2}{3}R^2 - \tfrac{1}{6}\,{}_2W_2) \quad\text{........(10)}.$$

373. It will hereafter be necessary to employ the transference formulae given in Chapter XI. § 227. Writing $\pi - \theta$ for θ in these formulae, and remembering that $P_n^m \{\cos(\pi - \theta)\} = (-)^{n-m} P_n^m (\cos\theta)$, and multiplying both sides of each equation by $\cos m\phi$, the formulae become

$$(n-m)!\,\frac{c^{n+1}\,{}_m w_n}{r^{2n+1}} = \sum_{s=0}^{s=\infty} \frac{(n+m+s)!}{(2m+s)!}\,\frac{{}_m W_{m+s}}{c^{m+s}} \;\ldots\ldots (11),$$

$$(n-m)!\,\frac{c^{n+1}\,{}_m W_n}{R^{2n+1}} = \sum_{s=0}^{s=\infty} \frac{(n+m+s)!}{(2m+s)!}\,\frac{{}_m w_{m+s}}{c^{m+s}} \;\ldots\ldots (12).$$

374. Let V, v be the potentials at an external point of the solids O, o respectively; and let V, v be divided into three parts V_1, V_2, V_3 and v_1, v_1, v_3 respectively. By (1), the condition that the free surfaces of the two masses of liquid should be equipotential surfaces, is that the equation

$$V_1 + V_2 + V_3 + v_1 + v_2 + v_3 + \Omega = \text{const.} \ldots\ldots\ldots (13),$$

should be satisfied at each of the free surfaces. Since the free surfaces are approximately spherical, each of the three v's will be of the form (7), and each of the three V's will be of a similar form with A and R written for a and r. Expressing the series (7) in terms of solid harmonics of positive degree instead of surface harmonics, it follows that (13) will be satisfied provided the following conditions are fulfilled.

(i) V_1 must consist of a series of zonal solid harmonics of the form (7) referred to the origin O, and v_1 must consist of a similar series referred to the origin o, such that when the expression $V_1 + v_1$ is transformed by means of (11) and (12) into two separate series of zonal solid harmonics referred to the two origins O and o respectively, the coefficients of all the harmonics must vanish except those of W_1 and w_1.

(ii) $V_2 + v_2$ must consist of two similar series of harmonics, such that when $V_2 + v_2$ is transformed into two separate series of zonal solid harmonics referred to O and o respectively, all the coefficients must vanish except those of W_1, W_2, w_1, w_2, and the coefficients of W_2, w_2 in $V_2 + v_2$ and the coefficients of W_1, w_1 in $V_1 + V_2 + v_1 + v_2$ must be determined so as to annul the terms involving these quantities in Ω.

(iii) $V_3 + v_3$ must consist of two series of tesseral solid harmonics ${}_2W_n$, ${}_2w_n$, such that when $V_3 + v_3$ is transformed into two

separate series referred to O and o respectively, all the coefficients must vanish except those of ${}_2W_2$, ${}_2w_2$, which must be determined so as to annul the terms involving these quantities in Ω.

The terms R^2 and r^2 in Ω need not be considered, for since the corresponding forces are symmetrical about each origin, they produce no departure from sphericity.

When we have determined the three quantities V_1+v_1, V_2+v_2, V_3+v_3 and the form of the boundary corresponding to each, the final result will be obtained by addition.

375. We shall now consider the potential V_1+v_1.

Let the equations of the two surfaces be

$$\frac{r}{a} = 1 + \left(\frac{A}{a}\right)^3 \sum_{n=2}^{n=\infty} \frac{2n+1}{2n-2}\left(\frac{a}{c}\right)^{n+1} h_n r^{-n} w_n \quad \ldots\ldots\ldots (14),$$

$$\frac{R}{A} = 1 + \left(\frac{a}{A}\right)^3 \sum_{n=2}^{n=\infty} \frac{2n+1}{2n-2}\left(\frac{A}{c}\right)^{n+1} H_n R^{-n} W_n \ldots\ldots (15),$$

where the h's and H's are unknown coefficients whose values are to be determined. Putting $\Gamma = A^2/c^2$, $\gamma = a^2/c^2$, it appears from (7) and (8) that

$$v_1 = \tfrac{4}{3}\pi a^2 \left(\frac{a}{r}\right) + \frac{2\pi A^3}{c} \sum_{k=2}^{k=\infty} \frac{h_k}{k-1}\left(\frac{a}{c}\right)^k \left(\frac{a}{r}\right)^{k+1} \frac{w_k}{r^k} \ldots\ldots (16),$$

$$V_1 = \frac{4\pi A^3}{3c}\left(\frac{c}{R}\right) + 2\pi a^3 \left(\frac{A}{c}\right)^3 \sum_{n=2}^{n=\infty} \frac{H_n \Gamma^{n-1}}{n-1}\frac{c^n W_n}{R^{2n+1}} \ldots\ldots (17).$$

Putting $m=0$ in (12), and transferring to o by the resulting formula, we obtain

$$V_1 = \frac{4\pi A^3}{3c} \sum_{k=0}^{k=\infty} \left(\frac{a}{c}\right)^k \frac{w_k}{a^k}$$
$$+ \frac{2\pi a^3}{c}\left(\frac{A}{c}\right)^3 \sum_{n=2}^{n=\infty} \frac{H_n \Gamma^{n-1}}{n-1} \sum_{k=0}^{k=\infty} \frac{(k+n)!}{k!\, n!}\left(\frac{a}{c}\right)^k \frac{w_k}{a^k},$$

the value of $v_1 + V_1$ is

$$\tfrac{4}{3}\pi a^2 \left(\frac{a}{r}\right) + \frac{4\pi A^3}{3c} \sum_{k=0}^{k=\infty} \left[\frac{3h_k}{2k-2}\left(\frac{a}{c}\right)^k \left(\frac{a}{r}\right)^{k+1} \frac{w_k}{r^k} + \frac{w_k}{c^k}\right.$$
$$\left. + \tfrac{3}{2}\left(\frac{a}{c}\right)^{3+k} \frac{w_k}{a^k} \sum_{n=2}^{n=\infty} \frac{(k+n)!}{n!\, k!}\frac{\Gamma^{n-1}}{n-1} H_n\right] \ldots\ldots (18).$$

This quantity is to vanish when r has the value given by (14) for all values of k except $k=1$. Since the squares and products of small quantities are to be neglected, we may put $r=a$ in the

term in square brackets, but in the first term we must give to a/r its full value; whence equating the coefficient of w_k to zero we obtain

$$-\frac{2k+1}{2k-2}h_k + \frac{3h_k}{2k-2} + 1 + \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{n=2}^{n=\infty} \frac{(n+k)!}{n!\,k!}\,\frac{\Gamma^{n-1}}{n-1}H_n = 0.$$

Therefore $$h_k = 1 + \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{n=2}^{n=\infty} \frac{(n+k)!}{n!\,k!}\,\frac{\Gamma^{n-1}}{n-1}H_n \text{ ....(19)},$$

and by symmetry

$$H_s = 1 + \tfrac{3}{2}\left(\frac{A}{c}\right)^3 \sum_{n=2}^{n=\infty} \frac{(n+s)!}{n!\,s!}\,\frac{\gamma^{n-1}}{n-1}h_n \text{ ....(20)}.$$

For the purpose of obtaining an approximate solution it will be sufficient to calculate the values of the H's and h's as far as c^{-5} only; we shall therefore require only the first terms of the two series, and we thus obtain

$$h_k = 1 + \tfrac{3}{2}\left(\frac{a}{c}\right)^5 \frac{(k+1)(k+2)}{2!} \text{ ....(21)},$$

$$H_k = 1 + \tfrac{3}{2}\left(\frac{A}{c}\right)^5 \frac{(k+1)(k+2)}{2!} \text{ ....(22)}.$$

Returning now to (18) we must determine the portion of the potential which involves harmonics of the first degree. From (16) it is seen that at the surface of o, v_1 contributes nothing; whence by (17) the portion of the potential is

$$u_1 = \frac{4\pi A^3}{3c^2}\left[1 + \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{n=2}^{n=\infty} \frac{n+1}{n-1}\Gamma^{n-1}H_n\right]w_1 \text{......(23)},$$

and when the origin is at O,

$$U_1 = \frac{4\pi a^3}{3c^2}\left[1 + \tfrac{3}{2}\left(\frac{A}{c}\right)^3 \sum_{n=2}^{n=\infty} \frac{n+1}{n-1}\gamma^{n-1}h_n\right]W_1 \text{......(24)}.$$

376. We must now consider the potential $V_2 + v_2$ due to the rotational terms, which are equal to $\frac{1}{6}\omega^2 W_2$ or $\frac{1}{6}\omega^2 w_2$ according as the origin is at O or o.

By Chapter XV., Ex. 14, if a spheroid of small ellipticity ϵ is rotating with angular velocity ω, then $\epsilon = 15\omega^2/16\pi$; let us therefore assume for the equations of the two masses of liquid

$$\frac{r}{a} = 1 + \tfrac{1}{3}\epsilon\frac{w_2}{r^2} + \left(\frac{A}{a}\right)^3 \sum_{n=2}^{n=\infty} \frac{2n+1}{2n-2}\left(\frac{a}{c}\right)^{n+1} l_n r^{-n} w_n \text{ ......(25)},$$

$$\frac{R}{A} = 1 + \tfrac{1}{3}\epsilon\frac{W_2}{R^2} + \left(\frac{a}{A}\right)^3 \sum_{n=2}^{n=\infty} \frac{2n+1}{2n-2}\left(\frac{A}{c}\right)^{n+1} L_n R^{-n} W_n \text{...(26)}.$$

By (7) the potential due to the inequality $\frac{1}{3}\epsilon r^{-2}w_2$ in (25) is equal to $\frac{4}{15}\pi\epsilon a^5 w_2/r^5$, whence proceeding as before, the potential v_2' due to the whole sphere a and its inequalities is

$$v_2' = \tfrac{4}{3}\pi a^2\left(\frac{a}{r}\right) + \tfrac{4}{15}\pi\epsilon w_2\left(\frac{a}{r}\right)^5 + \frac{2\pi A^3}{c}\sum_{k=2}^{k=\infty}\frac{l_k}{k-1}\left(\frac{a}{c}\right)^k\left(\frac{a}{r}\right)^{k+1}\frac{w_k}{r^k}\ \ldots(27),$$

and the potential V_2' due to the whole sphere A and its inequalities can be at once written down by symmetry. By (12) the value of V_2' when transferred to o is

$$V_2' = \frac{4\pi A^3}{3c}\sum_{k=0}^{k=\infty}\frac{w_k}{c^k} + \tfrac{2}{15}\pi\epsilon\frac{A^5}{c^3}\sum_{k=0}^{k=\infty}\frac{(k+1)(k+2)w_k}{c^k}$$

$$+\frac{2\pi a^3 A^3}{c^4}\sum_{n=2}^{n=\infty}\frac{L_n\Gamma^{n-1}}{n-1}\sum_{k=0}^{k=\infty}\frac{(k+n)!\,w_k}{k!\,n!\,c^k}\ \ldots\ldots\ldots\ldots\ldots\ldots(28).$$

Substituting the value of a/r from (25) in the first term of (27), we find that the value at the surface of the portion involving ϵ, added to the second term of (27)

$$= -\tfrac{4}{9}\pi\epsilon w_2 + \tfrac{4}{15}\pi\epsilon w_2 = -\tfrac{8}{45}\pi\epsilon w_2 = -\tfrac{1}{6}\omega^2 w_2.$$

This annuls the term $\frac{1}{6}\omega^2 w_2$ in the rotation potential; hence the value at a of the potential due to the inequalities of the two spheres minus the above mentioned term and the outstanding potentials of the first degree is,

$$V_2 + v_2 = -\frac{4\pi A^3}{3c}\sum_{n=2}^{n=\infty}\frac{(2n+1)\,l_n w_n}{(2n-2)\,c^n} + \frac{2\pi A^3}{c}\sum_{n=2}^{n=\infty}\frac{l_n}{n-1}\frac{w_n}{c_n}$$

$$+\frac{2\pi\epsilon A^5}{15c^3}\sum_{n=2}^{n=\infty}(n+1)(n+2)\,c^{-n}w_n$$

$$+\frac{2\pi A^3}{c}\left(\frac{a}{c}\right)^3\sum_{n=2}^{n=\infty}\sum_{k=2}^{k=\infty}\frac{(k+n)!}{k!\,n!}\frac{\Gamma^{k-1}}{k-1}L_k\frac{w_n}{c^n},$$

whence

$$-\frac{2n+1}{2n-2}l_n + \frac{3l_n}{2n-2} + \tfrac{1}{10}\epsilon\left(\frac{A}{c}\right)^2(n+1)(n+2)$$

$$+\tfrac{3}{2}\left(\frac{a}{c}\right)^3\sum_{k=2}^{k=\infty}\frac{\overline{k+n}\,!}{k!\,n}\frac{\Gamma^{k-1}}{k-1}L_k = 0,$$

or $$l_n = \tfrac{1}{10}\epsilon\left(\frac{A}{c}\right)^2(n+1)(n+2) + \tfrac{3}{2}\left(\frac{a}{c}\right)^3\sum_{k=2}^{k=\infty}\frac{\overline{k+n}\,!}{k!\,n!}\frac{\Gamma^{k-1}}{k-1}L_k.$$

Similarly

$$L_s = \tfrac{1}{10}\epsilon\left(\frac{a}{c}\right)^2(s+1)(s+2) + \tfrac{3}{2}\left(\frac{A}{c}\right)^3\sum_{k=2}^{k=\infty}\frac{\overline{k+s}\,!}{k!\,s!}\frac{\gamma^{k-1}}{k-1}l_k,$$

whence neglecting higher powers than c^{-5} we obtain

$$l_n = \tfrac{1}{10}\epsilon \left(\frac{A}{c}\right)^2 (n+1)(n+2) \ldots\ldots\ldots\ldots\ldots (29),$$

$$L_n = \tfrac{1}{10}\epsilon \left(\frac{a}{c}\right)^2 (n+1)(n+2) \ldots\ldots\ldots\ldots\ldots (30).$$

The outstanding potentials of the first degree are

$$u_2 = \tfrac{4}{5}\pi\epsilon A^5 w_1/c^4, \text{ and } U_2 = \tfrac{4}{5}\pi\epsilon a^5 W_1/c^4 \ldots\ldots\ldots (31).$$

377. Let us write Q_2, q_2 for ${}_2W_2$, ${}_2w_2$; and we have lastly to find the potential due to the rotational terms $-\frac{1}{12}\omega^2 q_2$ and $-\frac{1}{12}\omega^2 Q_2$ in Ω.

Let the equation of the free surfaces be

$$\frac{r}{a} = 1 - \tfrac{1}{6}\epsilon \frac{q_2}{r^2} - \left(\frac{A}{a}\right)^3 \sum_{n=2}^{n=\infty} \frac{2n+1}{2n-1}\left(\frac{a}{c}\right)^{n+1} m_n q_n r^{-n} \ldots\ldots (32),$$

$$\frac{R}{A} = 1 - \tfrac{1}{6}\epsilon \frac{Q_2}{R^2} - \left(\frac{a}{A}\right)^3 \sum_{n=2}^{n=\infty} \frac{2n+1}{2n-1}\left(\frac{A}{c}\right)^{n+1} M_n Q_n R^{-n} \ldots (33).$$

By (7) the potential due to the inequality $-\frac{1}{6}\epsilon q_2/r^2$ in (32) is $-2\pi\epsilon a^5 q_2/15r^5$; whence the potential v_3' of the mass o and its inequalities is

$$v_3' = \tfrac{4}{3}\pi a^2 \left(\frac{a}{r}\right) - \tfrac{2}{15}\pi\epsilon q_2 \left(\frac{a}{r}\right)^5 - \frac{2\pi A^3}{c} \sum_{n=2}^{n=\infty} \left(\frac{a}{c}\right)^n \left(\frac{a}{r}\right)^{n+1} \frac{m_n q_n}{(n-1) r^n} \; (34).$$

Whence at the surface, the value of the potential of the inequalities is

$$v_3 = \tfrac{4}{3}\pi a^2 \left\{\tfrac{1}{6}\epsilon q_2/a^2 + \left(\frac{A}{a}\right)^3 \sum_{n=2}^{n=\infty} \frac{2n+1}{2n-1}\frac{a m_n q_n}{c^{n+1}}\right\} - \tfrac{2}{15}\pi\epsilon q_2 \ldots (35),$$

and since $\qquad \tfrac{2}{9}\pi\epsilon q_2 - \tfrac{2}{15}\pi\epsilon q_2 = \tfrac{4}{45}\pi\epsilon q_2 = \tfrac{1}{12}\omega^2 q_2,$

the term $-\frac{1}{6}\epsilon q_2/r^2$ in (32) annuls the rotational term $\frac{1}{12}\omega^2 q_2$ in Ω.

The value referred to O of the potential of the inequalities of A is

$$V_3 = -\tfrac{2}{15}\pi\epsilon Q_2 \left(\frac{A}{R}\right)^5 - \frac{2\pi A^3 a^3}{c^4} \sum_{k=2}^{k=\infty} \frac{M_k \Gamma^{k-1}}{k-1} \frac{Q_k c^{k+1}}{R^{2k+1}},$$

and the value of this at the surface of o is

$$V_3 = -\tfrac{2}{15}\pi\epsilon A^5 c^{-3} \sum_{n=2}^{n=\infty} c^{-n} q_n$$

$$- \frac{2\pi A^3 a^3}{c^4} \sum_{k=2}^{k=\infty} \frac{M_k \Gamma^{k-1}}{k-1} \sum_{n=2}^{n=\infty} \frac{k+n\,!}{k-2\,!\; k+2\,!}\frac{q_n}{c^n} \ldots\ldots (36),$$

whence equating coefficients from (34), (35) and (36) we obtain

$$\frac{2n+1}{2n-1}m_n - \frac{3m_n}{2n-1} - \tfrac{1}{10}\pi\epsilon\left(\frac{A}{c}\right)^2$$
$$-\tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{k=2}^{k=\infty} \frac{k+n\,!}{k-2\,!\;n+2\,!}\frac{M_k\Gamma^{k-1}}{k-1} = 0.$$

Whence

$$m_n = \tfrac{1}{10}\epsilon\left(\frac{A}{c}\right)^2 + \tfrac{3}{2}\left(\frac{a}{c}\right)^3 \sum_{k=2}^{k=\infty} \frac{k+n\,!}{k-2\,!\;n+2\,!}\frac{\Gamma^{k-1}}{k-1}M_k.$$

Similarly

$$M_k = \tfrac{1}{10}\epsilon\left(\frac{a}{c}\right)^2 + \tfrac{3}{2}\left(\frac{A}{c}\right)^3 \sum_{s=2}^{s=\infty} \frac{s+k\,!}{s-2\,!\;k+2\,!}\frac{\gamma^{s-1}}{s-1}m_s.$$

Whence neglecting $(a/c)^6$ and $(A/c)^6$ we obtain

$$m_n = \tfrac{1}{10}\epsilon\,(A/c)^2,\quad M_n = \tfrac{1}{10}\epsilon\,(a/c)^2 \;\ldots\ldots\ldots\ldots (37).$$

378. In order to determine the **angular velocity**, we must equate to zero the sum of the harmonic terms of the first degree in Ω in (23) and (31); we thus obtain

$$-\omega^2 d + \tfrac{4}{3}\pi A^3/c^2 + \tfrac{4}{5}\pi\epsilon A^5/c^4 = 0,$$

or

$$-\omega^2 d + \tfrac{4}{3}\pi A^3/c^2 + \tfrac{3}{4}A^5\omega^2/c^4 = 0.$$

Similarly $$-\omega^2 D + \tfrac{4}{3}\pi a^3/c^2 + \tfrac{3}{4}a^5\omega^2/c^4 = 0.$$

Adding and remembering that $D + d = c$, we obtain

$$\omega^2\left\{1 - \tfrac{3}{4}(A^5 + a^5)/c^5\right\} = \tfrac{4}{3}\pi\,(A^3 + a^3)/c^3,$$

and since we neglect powers above $(a/c)^5$ we obtain

$$\omega^2 = \tfrac{4}{3}\pi\,(A^3 + a^3)/c^3 \;\ldots\ldots\ldots\ldots\ldots\ldots (38).$$

379. The object of the problem which we are considering is, to obtain the equations of the free surfaces of the two masses of liquid; this will be effected by adding the inequalities in equations (14), (25) and (32) to unity, and substituting the values of h_n, l_n, and m_n from (21), (29) and (37).

This will give us the equation of the boundary of the mass o.

Similarly by adding equations (15), (26) and (33) and substituting the values of H_n, L_n, and M_n from (22), (30) and (37), we shall obtain the form of the free surface of the mass O. We shall thus obtain

$$\frac{r}{a} = 1 + \tfrac{1}{6}\epsilon\,(2w_2 - q_2)/r^2 + \left(\frac{A}{a}\right)^3\left\{\tfrac{3}{2}\left(\frac{a}{c}\right)^3\frac{w_2}{r^2} + \tfrac{5}{4}\left(\frac{a}{c}\right)^4\frac{w_3}{r^3} + \tfrac{7}{6}\left(\frac{a}{c}\right)^5\frac{w_4}{r^4}\right\},$$

which expressed in terms of surface harmonics is

$$\frac{r}{a} = 1 + \tfrac{1}{6}\epsilon\,(2p_2 - p_2^{\,2}\cos 2\phi) + \left(\frac{A}{c}\right)^3 \left\{\tfrac{3}{2}p_2 + \tfrac{5}{4}\left(\frac{a}{c}\right)p_3 + \tfrac{7}{8}\left(\frac{a}{c}\right)^2 p_4\right\} \quad (39).$$

Similarly

$$\frac{R}{A} = 1 + \tfrac{1}{6}\epsilon\,(2P_2 - P_2^{\,2}\cos 2\phi) + \left(\frac{a}{c}\right)^3 \left\{\tfrac{3}{2}P_2 + \tfrac{5}{4}\left(\frac{A}{c}\right)P_3 + \tfrac{7}{8}\left(\frac{A}{c}\right)^2 P_4\right\} (40).$$

380. Prof. Darwin has entered into an elaborate series of numerical calculations for the purpose of ascertaining the forms of the two figures when they are in close proximity with one another; and has computed and drawn the figures which are shown in the accompanying diagrams.

Figures 1 and 2 show the form of the sections of the figures through and perpendicular to the axis of rotation when the masses are equal and nearly in contact, the constants being chosen so that $A = a$, $c/a = 2{\cdot}646$, $\omega^2/4\pi = {\cdot}038$, and h the moment of momentum $\propto {\cdot}472$. It will be observed that the section through the axis of rotation is considerably more elongated than the section perpendicular to that axis.

Figures 3 and 4 are particularly interesting. Here the masses are equal and $c/a = 2{\cdot}449$, $\omega^2/4\pi = {\cdot}0494$, $h \propto {\cdot}482$, and the masses partially overlap. Although two portions of matter cannot actually overlap so as to occupy the same portion of space, yet the continuity of figures of equilibrium leads to the conclusion that the two masses in this case constitute a single mass of liquid. The probable form of the free surface is shown by the dotted line connecting the two masses.

It will be observed that both the angular velocity and the moment of momentum of the system is greater in this case than in the preceding; it is therefore to be inferred that for a properly chosen moment of momentum, there exists a dumb-bell figure of equilibrium, and that when the ratio of the square of the angular velocity to the density is less than a certain quantity which lies between $4\pi \times {\cdot}0494$ and $4\pi \times {\cdot}038$, a single figure of equilibrium becomes impossible and the mass divides into two.

Figures 5 and 6 show the forms of the surfaces when the masses are unequal, the ratio of the larger mass to the smaller being 27. The free surfaces consist of two detached masses, and it is remarkable that the smaller mass has a very distinct furrow, which indicates a tendency for it to break up into two separate masses.

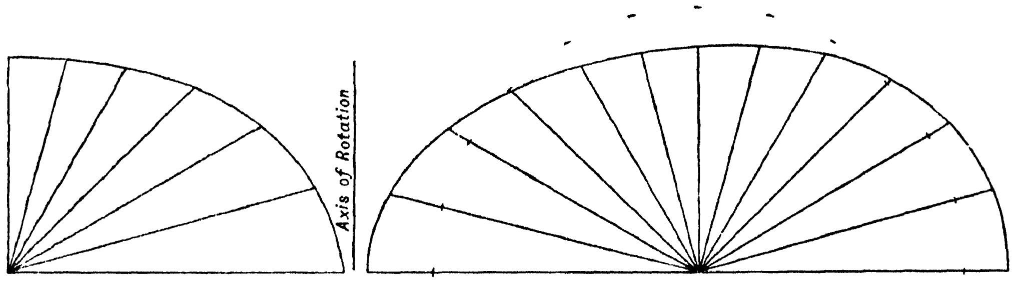


Fig.1

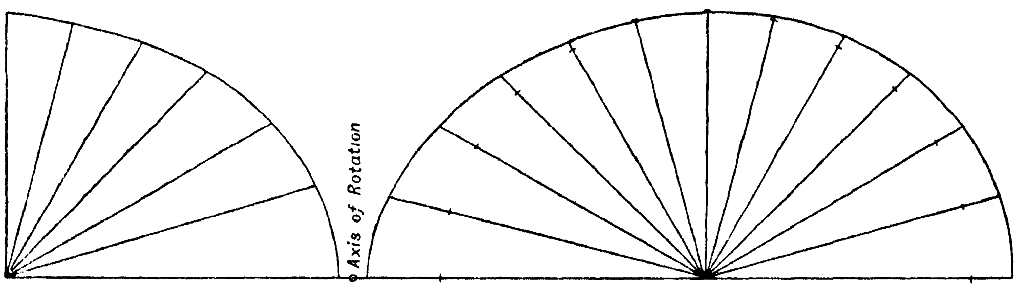


Fig.2

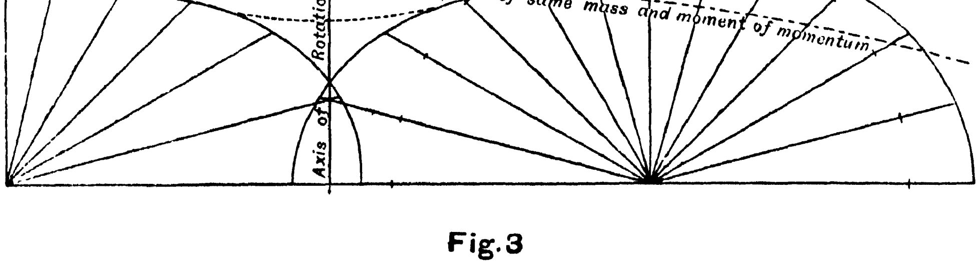


Fig.3

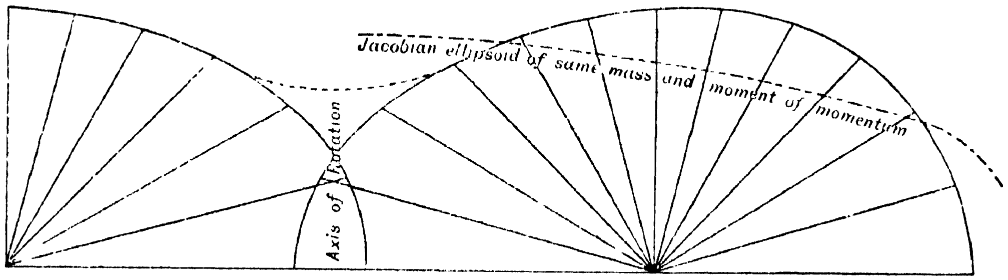


Fig. 4

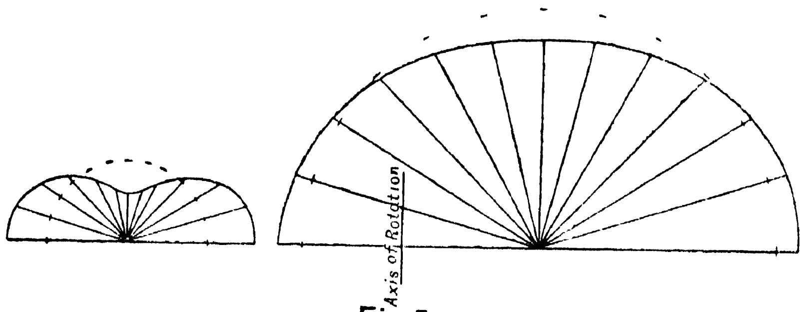


Fig. 5

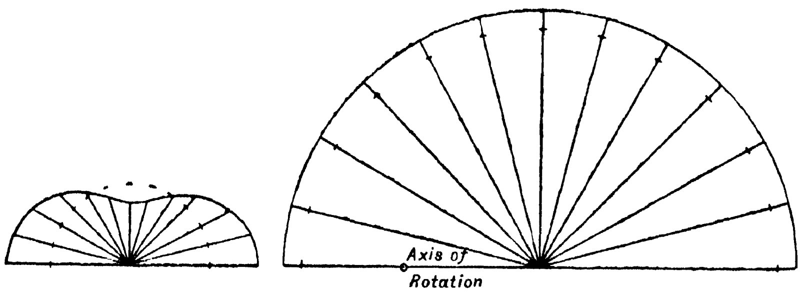


Fig. 6

381. Poincaré has shown by a difficult analytical process that when Jacobi's ellipsoid becomes moderately elongated, instability sets in by a furrowing of the ellipsoid along a line which lies in a plane perpendicular to the longest axis; and it is to be noticed that this furrow is not symmetrical with respect to the two ends. Assuming the correctness of this result, it would appear that there is a tendency to form a dumb-bell figure with two unequal bulbs.

MISCELLANEOUS EXAMPLES.

1. A vessel in the form of a regular polyhedron is filled with fluid, and revolves with given angular velocity around a vertical axis passing through its centre of gravity; if P denote the whole pressure on the surface of the vessel, S the whole surface, and Π the pressure at the centre of gravity, prove that $P - \Pi S$ is constant for every vertical axis.

2. Prove that the relative stream lines for liquid bounded by the hyperbolic cylinders $x(x-y) = a^2$, $y(x+y) = b^2$ are the quartic curves,

$$\{x(x-y) - a^2\}\{y(x+y) - b^2\} = \text{const.}$$

3. A right circular cylinder whose section is $r/a = 1 + f(\theta)$ where both $f(\theta)$ and $f'(\theta)$ are very small, is surrounded by an infinite liquid. If the cylinder have an angular velocity ω about its axis, prove that the velocity potential at any point of the liquid is

$$-\frac{a^3\omega r}{\pi}\int_0^{2\pi}\frac{f(\alpha)\sin(\alpha-\theta)\,d\alpha}{r^2 - 2ar\cos(\alpha-\theta) + a^2}.$$

4. A circular cylinder of radius a moves along the axis of x with velocity -1. Prove that the direction of motion of a particle of the fluid with respect to still water, is a tangent to the circle drawn through the particle and touching the axis of x at the point where the axis of the cylinder at the instant cuts this axis; and also that if ρ is the radius of curvature of the path of the particle relative to still water

$$a^2 = 4\rho\,(y - \tfrac{1}{2}b),$$

where b is a constant.

5. The resolved attractions of a body symmetrical about the axis of z are $f(z, \varpi)$ and $F(z, \varpi)$ respectively perpendicular and parallel to that axis. The equation of a solid of revolution is $\varpi f(z, \varpi) = a\varpi^2 + b$, where a and b are constants. Prove that if

this solid be made to move parallel to its axis in an infinite liquid, the stream lines are given by equating the left-hand side of this equation to a constant, and the velocity potential is $F(z, \varpi)$ multiplied by a constant.

When the moving solid is formed by two spheres intersecting at an angle π/n, find the velocity potential and current function by choosing as the attracting body a series of $2n-1$ particles, situated on the line joining the centres so that each is the image of the two adjacent particles, having their masses proportional to the cubes of their distances from any point on the intersection of the spheres, and being alternately attractive and repulsive.

6. Fluid moves irrotationally within an ellipsoidal cavity whose semi-axes are a, b, c in a vessel which turns freely about the axis of c. Show that the locus of points at which the pressure is the same as that at the centre is two planes, and that the pressure at any other point exceeds the pressure at the centre, by a quantity proportional to the product of its distances from these planes. Show also that each particle of fluid returns to the same place in the vessel after a time $T(a^2+b^2)/2ab$, where T is the time of a complete revolution of the vessel.

Find the place from which a drop of fluid may be removed without disturbing the motion.

Let an internal ellipsoid be described touching the cavity at the extremities of the axis of rotation, and having all its sections perpendicular to this axis similar to those of the cavity. If the mass of fluid within this ellipsoid be suddenly solidified and rigidly connected with the rotating vessel, find what change in the motion is produced.

7. Liquid is contained in a thin rigid ellipsoidal case, which is held in any position in contact with a smooth horizontal plane; if it is released, prove that the pressure on the table is instantaneously reduced in the ratio $1 : 1+P$, where

$$\tfrac{1}{5}Pp^2 = m^2n^2(b^2+c^2) + n^2l^2(c^2+a^2) + l^2m^2(a^2+b^2),$$

and p is the central perpendicular on the plane at striking, and l, m, n are its direction cosines referred to the principal axes of the ellipsoid.

Prove also that if it is dropped on to the plane, and has no rotation at striking, the kinetic energy is reduced by the impact in the ration $P : 1+P$.

CHAPTER XVII.

ON LIQUID WAVES.

382. THE different kinds of liquid waves may be classified as follows[1]:

I. *Oscillatory Waves*, which are the class of waves most commonly met with, and which consist of an elevation together with a companion hollow. They always appear in groups, and may either be stationary elevations or depressions, as in the case of a stream of running water, or may be propagated along the surface as at sea..

II. *The Wave of Translation or Solitary Wave*, which consists of a single wave travelling along the surface of the liquid. Its form may either be that of a solitary elevation or a solitary hollow, the former being called the positive wave, and the latter the negative wave. There is however an important difference between the two waves, since the positive wave possesses considerable permanence of form, being capable of propagation to great distances without suffering much degradation; whilst the negative wave is incapable of travelling any considerable distance without being broken up.

III. *Capillary Waves*, which are mainly produced by the surface tension of the liquid, and whose effect is insensible except near the surface of the liquid.

IV. *Sound Waves*, which in the case of liquids are due to the very slight changes which the density of a liquid under pressure experiences. They are insensible to sight, and the consideration of their properties belongs to the theory of sound rather than to hydrodynamics.

[1] Scott Russell, *Brit. Assoc. Rep. on Waves*, 1842—3.

The mathematical difficulties of the subject are so great, that no complete solution of any problem has as yet been obtained except the trochoidal waves considered in § 388, which were first discovered by Gerstner[1] in 1802, and afterwards independently by Rankine[2] in 1862; and we are therefore compelled to resort to approximate methods, which depend upon the assumption that the motion is sufficiently slow for it to be permissible to neglect the terms involving the squares and products of the velocities. The problem thus consists of (i) the determination of a velocity potential which satisfies Laplace's equation; (ii) the determination of the boundary conditions to be satisfied at the fixed boundaries of the liquid; (iii) the determination of the conditions to be satisfied in order that the free surface should be a surface of constant pressure, or in the case of two liquids which are in contact, that there should be no discontinuity of pressure at the surface of separation.

SECTION I.

Oscillatory Waves.

383. We shall first consider the waves propagated in a liquid of uniform depth h under the action of gravity.

Let the plane of the undisturbed surface be the plane of xy, let the axis of x be measured in the direction of propagation of the waves, and let the axis of z be measured vertically upwards.

Since the motion is supposed to be irrotational, the velocity potential satisfies the equation

$$\nabla^2\phi = 0 \qquad (1).$$

At the bottom of the liquid where $z = -h$,

$$d\phi/dz = 0 \qquad (2).$$

The pressure is determined by the equation

$$p/\rho + gz + \dot{\phi} + \tfrac{1}{2}q^2 = \text{const.} \qquad (3),$$

where q is the resultant velocity. At the free surface $\partial p/\partial t = 0$, or

$$\frac{dp}{dt} + u\frac{dp}{dx} + v\frac{dp}{dy} + w\frac{dp}{dz} = 0 \qquad (4).$$

[1] Theorie der Wellen, *Abhand. Kön. Böhmischen Gesel. Wiss.* 1802.

[2] *Phil. Trans.* 1863.

Also if η be the elevation of the free surface above the undisturbed surface, we must have

$$\dot{\eta} = d\phi/dz, \text{ when } z = \eta \qquad\qquad (5).$$

So far our equations have been exact; we shall now assume that the motion is so slow that the squares and products of the velocities may be neglected. Substituting the value of p from (3) in (4) and neglecting small quantities of the second order we obtain

$$\frac{d^2\phi}{dt^2} + g\frac{d\phi}{dz} = 0,$$

when $z = 0$. Since we are dealing with wave motion, ϕ must be an harmonic function of the time, whence if l be the length of the simple equivalent pendulum

$$l\frac{d^2\phi}{dt^2} + g\phi = 0,$$

and therefore $$ld\phi/dz = \phi, \text{ when } z = 0 \qquad\qquad (6).$$

Waves in Rectangular Canals.

384. When the motion is in two dimensions, we may suppose that the liquid is bounded by two parallel planes, which are at right angles to the crests of the waves. Hence the motion will be the same as that of waves propagated along a canal whose cross section is a rectangle.

Let λ be the length of the waves, U the velocity of propagation, h the depth of the canal. Since the motion is in two dimensions, we may assume

$$\phi = f(z)\cos(mx - nt),$$

where $m = 2\pi/\lambda$, $n = 2\pi U/\lambda$, $n^2 = g/l$. Substituting this value of ϕ in (1) we obtain

$$\frac{d^2f}{dz^2} - m^2f = 0;$$

the solution of which is

$$f = P\cosh mz + Q\sinh mz.$$

Equations (2) and (6) require that

$$P\sinh mh = Q\cosh mh$$

$$P = mlQ,$$

whence $$\phi = A \cosh m(z+h) \cos(mx - nt)$$

$$ml = \coth mh,$$

and $$U^2 = n^2/m^2 = g/m^2 l$$

$$= g\lambda/2\pi \,.\, \tanh 2\pi h/\lambda \quad \text{.................} \quad (7),$$

which determines the velocity of propagation.

Putting $4\pi h/\lambda = \mu$, we obtain

$$\frac{d}{d\mu} \log U^2 = -\mu^{-1} + \operatorname{cosech} \mu,$$

which is positive or negative, μ being supposed positive, according as

$$\mu > \text{or} < \sinh \mu > \text{or} < (\mu + \mu^3/3\,! + \ldots\ldots),$$

and is therefore negative. Hence U decreases as μ and therefore m increases, and therefore (7) cannot be satisfied for a given value of U by more than one value of m. Hence there is only one wave length which corresponds to a given velocity of propagation; also the velocity of propagation diminishes as the wave length increases.

385. When h/λ is small, $\tanh 2\pi h/\lambda = 2\pi h/\lambda$, and

$$U^2 = gh \quad \text{..............................} \quad (8),$$

which determines the velocity of propagation of long waves in shallow water.

When h/λ is large $\tanh 2\pi h/\lambda = 1$, and

$$U^2 = g\lambda/2\pi \quad \text{..........................} \quad (9),$$

which determines the velocity of propagation of deep sea waves.

386. At the free surface $z = \eta$, where η is the elevation; whence substituting the value of ϕ in (5) and suitably choosing the origin we obtain

$$\eta = -Amn^{-1} \sinh mh \sin(mx - nt).$$

Let (x, z) be the coordinates of an element of liquid when undisturbed, (ξ, ζ) its horizontal and vertical displacements, also let $x' = x + \xi$, $z' = z + \zeta$; then

$$\dot{\xi} = d\phi/dx' = -Am \cosh m(z'+h) \sin(mx' - nt)$$

$$\dot{\zeta} = d\phi/dz' = \quad Am \sinh m(z'+h) \cos(mx' - nt).$$

Since the displacement is small we may put $x=x'$, $z=z'$ as a first approximation, and we obtain

$$\xi = -a \cosh m(z+h) \cos(mx-nt)$$

$$\zeta = -a \sinh m(z+h) \sin(mx-nt),$$

where $Am/n=a$; whence the elements of liquid describe the ellipse

$$\xi^2/\cosh^2 m(z+h) + \zeta^2/\sinh^2 m(z+h) = a^2.$$

387. When the depth of the liquid is very great we may put $h=\infty$, and the hyperbolic functions must be replaced by exponential ones; we shall thus obtain

$$\phi = A\epsilon^{mz} \cos(mx-nt)$$

$$\eta = -Amn^{-1} \sin(mx-nt),$$

and the elements of liquid will describe the circles

$$\xi^2 + \zeta^2 = (Am/n)^2 \epsilon^{2mz}.$$

We shall consider the problem of deep sea waves at greater length in § 408.

Gerstner's Trochoidal Waves.

388. It was shown by Gerstner in 1802 and was rediscovered by Rankine, that there exists a certain form of trochoidal waves, which can be expressed in finite terms without resorting to methods of approximation.

Let the motion of the liquid be given by the equations

$$\left.\begin{aligned} x &= a + k^{-1} \epsilon^{-kb} \sin k(a+ct) \\ -z &= b + k^{-1} \epsilon^{-kb} \cos k(a+ct) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(10),$$

where k and c are absolute constants, and a and b are functions of the initial coordinates of the element of liquid whose coordinates at time t are (x, y).

The conditions of continuity require that the area of any elementary rectangle bounded by the curves a, b, $a+\delta a$, $b+\delta b$, should be constant throughout the motion, this requires that

$$\frac{d(x, z)}{d(a, b)} = A,$$

where A is a quantity which is independent of x, z or t. From (10) we obtain

$$\frac{d(x, z)}{d(a, b)} = \epsilon^{-2kb} - 1;$$

hence the conditions of continuity are satisfied.

The Lagrangian equations of motion are

$$\frac{d}{da}(p/\rho + gz) = -\ddot{x}\frac{dx}{da} - \ddot{z}\frac{dz}{da},$$

$$\frac{d}{db}(p/\rho + gz) = -\ddot{x}\frac{dx}{db} - \ddot{z}\frac{dz}{db},$$

which by (10) become

$$\frac{d}{da}(p/\rho + gz) = kc^2 \epsilon^{-kb} \sin k(a + ct),$$

$$\frac{d}{db}(p/\rho + gz) = kc^2 \epsilon^{-kb} \cos k(a + ct) - kc^2 \epsilon^{-2kb};$$

whence

$$p/\rho - g\{b + k^{-1}\epsilon^{-kb}\cos k(a + ct)\} = -c^2\epsilon^{-kb}\cos k(a + ct) + \tfrac{1}{2}c^2\epsilon^{-2kb} + C.$$

At the free surface p must be independent of t, whence

$$g = kc^2.$$

The wave length $\lambda = 2\pi/k$, and c is the velocity of propagation; hence $c = (g\lambda/2\pi)^{\frac{1}{2}}$, and is therefore equal to the velocity of propagation previously found for deep sea waves.

The pressure is given by the equation

$$\begin{aligned} p/\rho &= gb + \tfrac{1}{2}c^2\epsilon^{-2kb} + C \\ &= c^2(kb + \tfrac{1}{2}\epsilon^{-2kb}) + C, \end{aligned}$$

and therefore retains the same value at every point moving with the liquid. If therefore we put $b = \beta$ at the free surface, we obtain

$$p/\rho = c^2\{k(b - \beta) + \tfrac{1}{2}(\epsilon^{-2kb} - \epsilon^{-2k\beta})\},$$

which makes the pressure vanish at the free surface. The quantity b increases with $-z$, and therefore the wave disturbance decreases with the depth of the liquid.

The velocities of the liquid are

$$u = \dot{x} = c\epsilon^{-kb}\cos k(a + ct)$$

$$w = \dot{z} = c\epsilon^{-kb}\sin k(a + ct),$$

from which it can be shown that a velocity potential does not exist. In fact

$$\left(\frac{dw}{dx} - \frac{du}{dz}\right)\frac{d(x, z)}{d(a, b)} = w_a z_b - w_b z_a + u_a x_b - u_b x_a,$$

where the suffixes denote partial differentiation; whence if ω be the molecular relation

$$\omega = kc\epsilon^{-2kb}/(\epsilon^{-2kb} - 1).$$

The motion is therefore rotational, and therefore waves of this description could not be generated in a frictionless liquid which is under the action of natural forces.

Waves at the Surface of Separation of Two Liquids.

389. Let us first suppose that two liquids of different densities (such as water and mercury) are resting upon one another, which are in repose except for the disturbance produced by the wave motion, and which are confined between two planes parallel to their surface of separation. Let ρ, ρ' be the densities of the lower and upper liquids respectively, h, h' their depths, and let the origin be taken in the surface of separation when in repose.

In the lower liquid let

$$\phi = A \cosh m(z + h) \cos(mx - nt) \quad\ldots\ldots\ldots\ldots(11),$$

and in the upper let

$$\phi' = A' \cosh m(z - h') \cos(mx - nt) \quad\ldots\ldots\ldots\ldots(12),$$

also let

$$\eta = a \sin(mx - nt),$$

be the equation of the surface of separation. At this surface, the condition that the two liquids should remain in contact requires that

$$d\eta/dt = d\phi/dz = d\phi'/dz, \text{ when } z = 0.$$

Whence $\quad -na = mA \sinh mh = -mA' \sinh mh'.$

If δp, $\delta p'$ be the increments of the pressure due to the wave motion just below and just above the surface of separation, then

$$\delta p + g\rho\eta + \rho d\phi/dt = 0,$$

and

$$\delta p' + g\rho'\eta + \rho' d\phi'/dt = 0,$$

and since $\delta p = \delta p'$, we obtain

$$\begin{aligned} g(\rho - \rho')\eta &= -\rho d\phi/dt + \rho' d\phi'/dt \\ &= n(-A\rho \cosh mh + A'\rho' \cosh mh') \sin(mx - nt) \\ &= (\rho \coth mh + \rho' \coth mh') n^2\eta/m, \end{aligned}$$

whence

$$U^2 = (n/m)^2 = \frac{g(\rho - \rho')}{m(\rho \coth mh + \rho' \coth mh')},$$

where $m = 2\pi/\lambda$.

390. When λ is small compared with h and h', then mh, mh' are large, and coth mh and coth mh' may be replaced by unity; we thus obtain

$$U^2 = g(\rho - \rho')/m(\rho + \rho').$$

If $\rho' > \rho$, U^2 is negative and therefore n is imaginary; hence if the upper liquid is denser than the lower the motion cannot be represented by a periodic term in t, and is therefore unstable.

If the density of the upper liquid is small compared with that of the lower, we have approximately

$$U^2 = gm^{-1}(1 - 2\rho'/\rho).$$

If the liquid is water in contact with air, $\rho'/\rho = \cdot 00122$, hence if the air is treated as an incompressible fluid

$$U^2 = \cdot 99756 \times gm^{-1}.$$

391. Secondly, let us suppose that the upper liquid is moving with velocity V', and the lower with velocity V; then we may put

$$\phi = Vx + A \cosh m(z + h) \cos(mx - nt)$$

$$\phi' = V'x + A' \cosh m(z - h') \cos(mx - nt).$$

Let the equation of the surface of separation be

$$F = \eta - a \sin(mx - nt) = 0.$$

Then in both liquids F must be a bounding surface, and therefore when $z = 0$,

$$\frac{dF}{dt} + \frac{d\phi}{dx}\frac{dF}{dx} + \frac{dF}{d\eta}\frac{d\phi}{dz} = 0,$$

$$\frac{dF}{dt} + \frac{d\phi'}{dx}\frac{dF}{dx} + \frac{dF}{d\eta}\frac{d\phi'}{dz} = 0.$$

Whence $$an - mVa + mA \sinh mh = 0$$

$$an - mV'a - mA' \sinh mh' = 0.$$

Hence if $U = n/m$ be the velocity of propagation,

$$A \sinh mh = a(V - U)$$
$$A' \sinh mh' = -a(V' - U).$$

If δp, $\delta p'$ be the increments of pressure at the surface of separation due to the wave motion,

$$\delta p/\rho + g\eta + d\phi/dt + \tfrac{1}{2}\{V - Am \cosh mh \cos(mx - nt)\}^2 = \tfrac{1}{2}V^2,$$
$$\delta p'/\rho' + g\eta + d\phi'/dt + \tfrac{1}{2}\{V' - A'm \cosh mh' \cos(mx - nt)\}^2 = \tfrac{1}{2}V'^2.$$

Therefore since $\delta p = \delta p'$,

$$ag(\rho - \rho') = Am\rho(V - U)\cosh mh - A'm\rho'(V' - U)\cosh mh'$$

or $$g(\rho - \rho') = m\rho(V - U)^2 \coth mh + m\rho'(V' - U)^2 \coth mh',$$

which determines U.

Waves in Canals with Sloping Sides.

392. In all the preceding sections the motion considered has been in two dimensions, and the results are therefore applicable either to straight crested waves in an unlimited ocean, or to waves in a canal whose cross section is a rectangle. We shall now consider some cases of three-dimensional motion.

We shall first discuss the case of waves propagated along a straight canal of uniform section, whose sides are two planes inclined at an angle $\frac{1}{4}\pi$ to the horizon.

Let h be the greatest depth of the canal, and let the origin be taken in the line of intersection of the two sides. The equations of the two sides of the canal are $y \pm z = 0$, and the boundary conditions are

$$d\phi/dy - d\phi/dz = 0 \text{ when } y - z = 0$$
$$d\phi/dy + d\phi/dz = 0 \text{ when } y + z = 0.$$

The equation of continuity and the boundary conditions will be satisfied if

$$\phi = A \cosh my \cosh mz \cos \sqrt{2}(mx - nt).$$

At the free surface where $z = h$, we must have

$$l\,d\phi/dz = \phi,$$

for all values of x and y, whence

$$ml = \coth mh,$$

and therefore $U^2 = n^2/m^2 = g/lm^2 = gm^{-1} \tanh mh,$

$$= (g\lambda/\pi\sqrt{2}) \tanh \pi h\sqrt{2}/\lambda.$$

The free surface is determined by the equation

$$d\eta/dt = d\phi/dz = mA \sinh mh \cosh my \cos \sqrt{2}\,(mx - nt),$$

whence $\eta = -(mA/n\sqrt{2}) \sinh mh \cosh my \sin \sqrt{2}\,(mx - nt).$

These results are due to Prof. Kelland.

The equation of continuity and the boundary conditions will also be satisfied by assuming

$$\phi = B \sinh my \sinh mz \sin \sqrt{2}\,(mx - nt),$$

in which case we should have

$$U^2 = (g\lambda/\pi\sqrt{2}) \coth \pi h\sqrt{2}/\lambda,$$

$$\eta = (mA/n\sqrt{2}) \cosh mh \sinh my \cos \sqrt{2}\,(mx - nt).$$

393. Kelland[1] also obtained the solution for progressive waves whose crests are perpendicular to a shore whose inclination to the horizon is $\frac{1}{4}\pi$, and which are moving parallel to the shore. This solution has been generalized by Prof. Stokes[2] for a shore sloping at any angle α.

Let the origin be taken in the line of intersection of the shore with the undisturbed surface; then the equation of the shore will be

$$y \sin \alpha + z \cos \alpha = 0,$$

and the boundary condition is

$$\frac{d\phi}{dy} \sin \alpha + \frac{d\phi}{dz} \cos \alpha = 0,$$

which is satisfied if

$$\phi = A \exp \{-m\,(y \cos \alpha - z \sin \alpha)\} \cos (mx - nt).$$

Whence $$ml \sin \alpha = 1,$$

and $$U^2 = (g\lambda/2\pi) \sin \alpha.$$

394. If we attempt to determine the solution for progressive waves along a canal whose sides slope at an angle $\frac{1}{6}\pi$ to the horizon, by assuming that $\phi = F(y, z) \cos \sqrt{2}\,(mx - nt)$, it will be found that the period equation has only one real root, viz. $mh = 0$,

[1] *Trans. Roy. Soc. Edin.* vol. xv. p. 121.

[2] *Brit. Assoc. Rep.* Hydrodynamics, 1846.

repeated four times. Hence it follows that progressive waves in a canal of this form are unstable; we must therefore assume

$$\phi = \Phi \cosh \sqrt{2}\,(mx - nt).$$

The boundary conditions are

$$d\Phi/dy = d\Phi/dz, \text{ when } y = z\sqrt{3},$$
$$d\Phi/dy = -\,d\Phi/dz, \text{ when } y = -\,z\sqrt{3}.$$

These equations together with the equation of continuity will be satisfied by assuming

$$\Phi = \sin m\,(z - \alpha) \cos my + \sin \tfrac{1}{2}m\,\{(\sqrt{3} - 1)\,z - 2\alpha\} \cos \tfrac{1}{2}m\,(\sqrt{3} + 1)\,y$$
$$- \sin \tfrac{1}{2}m\,\{(\sqrt{3} + 1)\,z + 2\alpha\} \cos \tfrac{1}{2}m\,(\sqrt{3} - 1)\,y.$$

Substituting this value of Φ in (6) and putting $m\,(h - \alpha) = \gamma$, we obtain the following equations:

$$ml = \tan\gamma = (\sqrt{3} + 1) \tan\{\gamma - \tfrac{1}{2}\,(3 - \sqrt{3})\,mh\}$$
$$= (\sqrt{3} - 1) \tan\{\tfrac{1}{2}\,(3 + \sqrt{3})\,mh - \gamma\}.$$

From these equations we find that $\tan\gamma$ is an harmonic mean between $\tan\frac{1}{2}\,(3 - \sqrt{3})\,mh$ and $\tan\frac{1}{2}\,(3 + \sqrt{3})\,mh$, which determines γ and therefore α in terms of mh; and on eliminating γ we shall find that the period equation is

$$(2 - \sqrt{3}) \cos(3 + \sqrt{3})\,mh + (2 + \sqrt{3}) \cos(3 - \sqrt{3})\,mh - \cos 2mh\sqrt{3} = 3,$$

which is an equation with an infinite number of real roots.

Since wave motion is stable when the sides of the canal are inclined at an angle $\frac{1}{4}\pi$ to the horizon, and unstable when they are inclined at an angle $\frac{1}{6}\pi$, it follows that there must be some inclination lying between $\frac{1}{4}\pi$ and $\frac{1}{6}\pi$ which forms the limit between stability and instability. The value of this angle has not apparently been determined.

Standing Waves across a Canal.

395. If liquid is contained in a straight canal whose sides are inclined at any angle α to the horizon, and if the free surface is either displaced in such a manner that its equation is $\eta = F(y)$, where y is measured across the canal, and the liquid is then left to itself; or if a velocity $f(y)$ is communicated to every point of the free surface, after which the liquid is left to itself; the subsequent motion of the liquid, if periodic, will consist of oscillations composed

of waves whose crests are parallel to the sides of the canal. Such oscillations are called *standing waves*, and the theory of them has been investigated by Kirchhoff[1] and Greenhill[2].

When the sides of the canal are inclined at an angle $\frac{1}{4}\pi$ to the horizon, the boundary conditions are

$$\left.\begin{aligned} d\phi/dy - d\phi/dz = 0 \text{ when } y - z = 0 \\ d\phi/dy + d\phi/dz = 0 \text{ when } y + z = 0 \end{aligned}\right\} \quad \text{..........} (13).$$

We can at once obtain an algebraic solution, by supposing that the free surface is initially plane.

Let $\eta = ay$, $\dot{\eta} = 0$ initially. The equation of continuity and (13) are satisfied if

$$\phi = Ayz \sin nt.$$

From (6) we obtain $l = h$; also

$$\dot{\eta} = d\phi/dz = Ay \sin nt.$$

Whence $$\eta = -An^{-1}y \cos nt,$$

and therefore $$\phi = -anyz \sin nt.$$

The value of the current function ψ is

$$\psi = \tfrac{1}{2}an(y^2 - z^2) \sin nt,$$

which shows that the stream lines are rectangular hyperbolas.

396. The equation of continuity and (13) are also satisfied if

$$\phi = \tfrac{1}{2}A\{\cos m(y + \iota z) \pm \cos m(y - \iota z) + \cos m(z + \iota y) \pm \cos m(z - \iota y)\} (\cos \textit{or} \sin) nt.$$

Taking the upper sign, and putting $mh = p$, we obtain from (6)

$$m\dot{l}(\cos my \sinh p - \cosh my \sin p) = \cos my \cosh p + \cosh my \cos p.$$

Since this equation is true for all values of y, we must have

$$ml = \coth p = -\cot p \quad \text{....................}(14).$$

Similarly if we had taken the lower sign, we should have obtained

$$ml = \tanh p = \tan p \quad \text{....................} (15).$$

Both the period equations (14) and (15) are included in the single equation

$$\cos 2p \cosh 2p = 1,$$

[1] Ueber Stehende Schwinzungen einer schweren Flüssigkeit, *Gesam. Abhand.* vol. II.

[2] *Amer. Jour. of Math.* vol. IX. p. 62.

which is the period equation for the lateral vibrations of a bar. This equation is discussed in Lord Rayleigh's *Theory of Sound*, vol. I. p. 219, and it is there shown that it has an infinite number of real roots.

397. In order to find the solution for standing waves parallel to a shore which slopes at an angle $\frac{1}{4}\pi$, let

$$\phi = A\,\{\epsilon^{m(z-\iota y)} + \epsilon^{-m(y-\iota z)}\}\,(\cos\ or\ \sin)\,nt,$$

the origin being in the line of intersection of the undisturbed surface with the shore, and y being measured from the shore.

This value of ϕ satisfies the boundary condition

$$d\phi/dy + d\phi/dz = 0 \text{ when } z = -y.$$

If we take the real part of this expression alone, it will be found impossible to satisfy (6), but if we add together the real and imaginary parts we obtain

$$\phi = A\,\{\epsilon^{mz}(\cos my - \sin my + \epsilon^{-my}(\cos mz + \sin mz)\}\,(\cos\ or\ \sin)\,nt,$$

and (6) gives $$ml = 1.$$

Whence $$U^2 = g\lambda/2\pi.$$

398. The corresponding solutions for standing waves across a canal whose sides are inclined at an angle $\frac{1}{6}\pi$ to the horizon, have also been obtained by Kirchhoff. In this case we can obtain an algebraic solution by supposing that the initial form of the free surface is the parabolic cylinder

$$\eta = a\,(h^2 - y^2),$$

where h is the depth of the liquid, and the origin is a point in the intersection of the sides.

The equation of continuity is satisfied if

$$\phi = A\Phi \sin nt,$$

where $$\Phi = z^3 - 3y^2z + 2h^3,$$

and the corresponding current function is

$$\Psi = y\,(z\surd 3 - y)\,(z\surd 3 + y),$$

which vanishes when $y = \pm z\surd 3$, so that the boundary conditions are satisfied.

At the free surface $z = h$, and

$$\frac{d\Phi}{dz} = 3h^2 - 3y^2 = \Phi/h,$$

and therefore $l = h$. Also

$$\dot{\eta} = 3A\,(h^2 - y^2)\sin nt,$$

therefore $$\eta = -\,3An^{-1}(h^2 - y^2)\cos nt,$$

which shows that the initial form of the free surface is a parabolic cylinder.

For the solution in the general case, we must refer the reader to Prof. Greenhill's article on Waves[1].

Waves in a Cylinder[2].

399. The equation of continuity referred to cylindrical coordinates ϖ, θ, z is

$$\frac{d^2\phi}{d\varpi^2} + \frac{1}{\varpi}\frac{d\phi}{d\varpi} + \frac{1}{\varpi^2}\frac{d^2\phi}{d\theta^2} + \frac{d^2\phi}{dz^2} = 0 \quad\text{..............(16).}$$

If h be the depth of the liquid, the surface conditions are

$$d\phi/dz = 0 \text{ when } z = -h \quad\text{..................(17),}$$

$$ld\phi/dz = \phi \text{ when } z = 0 \quad\text{.....................(18).}$$

In order to satisfy (16), assume

$$\phi = AF(\varpi)\sin n\theta \cosh(kz + \beta)\cos pt.$$

Substituting in (16) we obtain

$$\frac{d^2F}{d\varpi^2} + \frac{1}{\varpi}\frac{dF}{d\varpi} - \frac{n^2F}{\varpi^2} + k^2F = 0 \quad\text{..............(19),}$$

whence $$F = J_n(k\varpi).$$

If a be the radius of the cylinder, $d\phi/d\varpi = 0$ when $r = a$, whence

$$J'_n(ka) = 0 \quad\text{..........................(20),}$$

and the different values of k are the roots of (20), which can be shown to be all real.

The value of n will depend upon the particular problem under consideration. If the motion is symmetrical about the origin, $n = 0$; if on the other hand the liquid is contained within a sector of angle 2α where $\alpha < \frac{1}{2}\pi$, and if the line bisecting the angle of the

[1] *American Journal of Mathematics*, vol. IX. p. 62.

[2] Lord Rayleigh, "On Waves," *Phil. Mag.* April, 1876.

sector is taken as the initial line, we must have $d\phi/d\theta = 0$ when $\theta = \pm\alpha$, whence $n = (2m+1)\pi/2\alpha$ where m is a positive integer.

From (17) we obtain $\beta = kh$; and from (18) we find

$$kl = \coth kh,$$

whence
$$p^2 = gk \tanh kh.$$

400. Let us now suppose that the liquid is initially at rest, and that the free surface is displaced so that its initial form is

$$\eta = \varpi \cos\theta.$$

Then
$$\phi = \Sigma A J_1(k\varpi)\cos\theta \cosh k(z+h)\sin pt,$$

and
$$d\phi/dz = d\eta/dt = \Sigma k A J_1(k\varpi)\sinh kh \cos\theta \sin pt,$$

and
$$\eta = -\Sigma k p^{-1} A J_1(k\varpi)\sinh kh \cos\theta \cos pt \quad\text{.........(21).}$$

Initially $\eta = \varpi\cos\theta$, therefore

$$\varpi = -\Sigma k p^{-1} A \sinh kh\, J_1(k\varpi),$$

and putting $Akp^{-1}\sinh kh = -B$, we obtain

$$\varpi = \Sigma B J_1(k\varpi),$$
$$\eta = \Sigma B J_1(k\varpi)\cos\theta\cos pt.$$

Let $I_1 = J_1(k'\varpi)$, then, if the accents denote differentiation with respect to ϖ,

$$\varpi^2 J_1'' + \varpi J_1' + (k^2\varpi^2 - 1) J_1 = 0,$$
$$\varpi^2 I_1'' + \varpi I_1' + (k'^2\varpi^2 - 1) I_1 = 0,$$

whence
$$(k'^2 - k^2)\int_0^a \varpi I_1 J_1 d\varpi + a(I_1' J_1 - I_1 J_1')_a = 0.$$

Since $I'_1(k'a)$ and $J'_1(ka) = 0$, the integral must vanish if k and k' are different; to find the value of the integral when $k = k'$, let $k' = k + \delta k$, then

$$2k\delta k\int_0^a \varpi J_1^2(k\varpi)\,d\varpi + a\left(J_1\frac{dJ_1'}{dk} - J_1'\frac{dJ_1}{dk}\right)\delta k = 0,$$

therefore
$$\int_0^a \varpi J_1^2(k\varpi)\,d\varpi = -\frac{a^2}{2k^2}J_1 J_1''.$$

Hence
$$-Ba^2 J_1 J_1''/2k^2 = \int_0^a \varpi^2 J_1(k\varpi)\,d\varpi.$$

But $$\varpi \frac{d}{d\varpi}(\varpi J_1') + (k^2\varpi^2 - 1) J_1 = 0,$$

whence $$\int_0^a (k^2\varpi^2 - 1) J_1 d\varpi - aJ_1 + \int_0^a J_1 d\varpi = 0,$$

and therefore

$$B = -\frac{2}{aJ_1''} = \frac{2a}{(k^2a^2 - 1) J_1(ka)},$$

and $$\eta = \Sigma \frac{2aJ_1(k\varpi)\cos\theta\cos pt}{(k^2a^2 - 1) J_1(ka)},$$

which determines the form of the free surface at any subsequent time.

Waves in Hyperboloids and Cones.

401. If we put

$$\phi = z\varpi^n \sin n\theta \cos pt \text{..................(22)},$$

(16) is satisfied; also at the free surface where $z = h$,

$$hd\phi/dz = \phi$$

so that $l = h$.

Let us suppose that the vessel which contains liquid having this motion is one of revolution; in order to determine its shape, we have along a meridian section

$$\frac{d\phi}{d\varpi} dz = \frac{d\phi}{dz} d\varpi,$$

or $$nzdz = \varpi d\varpi$$

by (22); whence integrating

$$nz^2 = \varpi^2 + C.$$

The containing vessel is therefore a hyperboloid of revolution, including as a particular case a cone of semi-vertical angle $\tan^{-1}\sqrt{n}$.

Long Waves in Shallow Water.

402. In the theory of long waves it is assumed that the length of the waves is so great in proportion to the depth of the water, that the vertical component of the velocity can be neglected, and the horizontal component is uniform across each section of the canal. In § 385 we saw that if the depth is small compared with

the wave length then $U^2 = gh$, provided the square of the velocity is neglected. We shall now examine this result in connection with the above-mentioned assumption.

Let the motion be made steady by impressing on the whole liquid a velocity equal and opposite to the velocity of propagation of the waves. Let η be the elevation of the liquid above the undisturbed surface; U, u the velocities corresponding to h and $h + \eta$ respectively. The equation of continuity gives

$$u = hU/(h + \eta),$$

whence $$U^2 - u^2 = U^2 (2h\eta + \eta^2)/(h + \eta)^2.$$

If δp be the excess of pressure due to the wave motion

$$\delta p = \left\{\frac{U^2 (2h + \eta)}{2 (h + \eta)^2} - g\right\} \rho\eta.$$

When η/h is very small the quantity in brackets is $U^2/h - g$; whence if $U^2 = gh$, the change of pressure at a height $h + \eta$ vanishes to a first approximation and therefore a free surface is possible.

If the condition $U^2 = gh$ is satisfied, the change of pressure to a second approximation is

$$\delta p = - 3g\rho\eta^2/2h,$$

which shows that the pressure is defective at all parts of the wave at which η differs from zero. *Unless therefore η^2 can be neglected, it is impossible to satisfy the condition of a free surface for a stationary long wave;—in other words, it is impossible for a long wave of finite height to be propagated in still water without change of type.* If however η be everywhere positive a better result can be obtained with a somewhat increased value of U; and if η be everywhere negative, with a diminished value. We therefore infer that positive waves travel with a somewhat higher, and negative waves with a somewhat lower velocity than that due to half the undisturbed depth[1].

403. The theory of long waves in a canal may be investigated analytically as follows[2].

Let the origin be in the bottom of the liquid, h the undisturbed depth, η the elevation; and let x be the abscissa of an element of liquid when undisturbed, ξ the horizontal displacement. The

[1] Lord Rayleigh, "On Waves," *Phil. Mag.* April, 1876.

[2] Airy, "Tides and Waves," *Encyc. Met.*

quantity of liquid originally between the planes x and $x+dx$ is hdx; at the end of an interval t, the breadth of this stratum is $dx\,(1+d\xi/dx)$, and its height is $h+\eta$, whence the equation of continuity is

$$(1+d\xi/dx)\,(h+\eta)=h \quad\text{.................(23).}$$

Let us now investigate the motion of a column of liquid contained between the planes whose original distance was dx; and let us suppose that in addition to gravity, small horizontal and vertical disturbing forces X and Y act. Since the vertical acceleration is neglected, the pressure will be equal to the hydrostatic pressure due to a column of liquid of height $h+\eta$, whence

$$p=g\rho\,(h+\eta-y)+\rho\int_y^{h+\eta} Y dy \quad\text{...........(24).}$$

The equation of motion of the stratum is

$$\rho h\frac{d^2\xi}{dt^2}=-\frac{dp}{dx}(h+\eta)+X\rho h \text{.................(25).}$$

Now from (24),

$$\frac{dp}{dx}=g\rho\frac{d\eta}{dx}+\rho Y\frac{d\eta}{dx}+\rho\int_y^{h+\eta}\frac{dY}{dx}dy \text{...........(26);}$$

also in most problems to which the theory applies the last two terms on the right-hand side of (26) are very much smaller than the first, and may therefore be neglected, whence (25) becomes

$$h\frac{d^2\xi}{dt^2}=-g\,(h+\eta)\frac{d\eta}{dx}+Xh.$$

Substituting the value of η from (23) we obtain

$$\frac{d^2\xi}{dt^2}=gh\frac{d^2\xi}{dx^2}\left(1+\frac{d\xi}{dx}\right)^{-3}+X \text{ .... (27).}$$

For a first approximation, we may neglect squares and products of small quantities, and (23) and (27) respectively become

$$\eta/h=-d\xi/dx \text{..............................(28),}$$

$$\frac{d^2\xi}{dt^2}=gh\frac{d^2\xi}{dx^2}+X \quad\text{.......................(29).}$$

If $X=0$, the form of (29) shows that the velocity of propagation is equal to $(gh)^{\frac{1}{2}}$.

Stationary Waves in Flowing Water[1].

404. Let us suppose that water is flowing uniformly along a straight canal with vertical sides, and that between two points A and B there are small inequalities, and that beyond these points the bottom is perfectly level. Let a be the depth, u the velocity, p the *mean* pressure beyond A; b the depth, v the velocity, and q the *mean* pressure beyond B: also let f be the difference of levels of the bottom at A and B.

The total energy of the liquid per unit of the canal's length and breadth, at points beyond B is

$$\tfrac{1}{2}v^2 b + g\int_0^b y\,dy + w = \tfrac{1}{2}(v^2 + gb)\,b + w,$$

where w is the wave energy, and the density of the liquid is taken as unity. At very great distances beyond B the wave motion will have subsided and w will be zero.

The equation of continuity is

$$au = bv = M \qquad (30).$$

The dynamical equation is found from the consideration that the difference between the work done by the pressure p upon the volume of water entering at A, and the work done by the pressure q at B upon an equal volume of water passing away at B, is equal to the difference between the energy which passes away at B, and the energy which enters at A. Whence

$$pau - qbv = (\tfrac{1}{2}v^2 b + \tfrac{1}{2}gb^2 + w)\,v - (\tfrac{1}{2}u^2 a + g\int_f^{a+f} y\,dy)\,u,$$

which by (30) becomes,

$$p - q = \tfrac{1}{2}v^2 + \tfrac{1}{2}gb + w/b - \tfrac{1}{2}u^2 - g\,(f + \tfrac{1}{2}a) \qquad (31).$$

Now p and q are the *mean* pressures, and therefore since the pressure at the free surface is zero,

$$p = \tfrac{1}{2}ga, \quad q = \tfrac{1}{2}gb + w'/b,$$

where w' denotes a quantity depending on the wave disturbance; whence (31) becomes

$$\tfrac{1}{2}M^2(a^2 - b^2)/a^2b^2 - g\,(a - b + f) + (w - w')/b = 0 \qquad (32).$$

If we put

$$D^3 = 2a^2b^2/(a + b), \quad M = VD;$$

[1] Sir W. Thomson, *Phil. Mag.* (5) vol. XXII. p. 353.

D will denote a mean depth intermediate between a and b, and approximately equal to their arithmetic mean when their difference is small in comparison with either; and V will similarly denote a corresponding mean velocity of flow. We thus obtain from (32)

$$b - a = \frac{f - (w - w')/gb}{1 - V^2/gD}.$$

If $b - a$ were exactly equal to f, and there were no disturbance of the water beyond B, the mean level of the water would be the same at great distances beyond A and B; but if this is not the case, there will be a rise or fall of level, determined by the formula

$$y = b - a - f = \frac{V^2 f/gD + (w - w')/gb}{1 - V^2/gD}.$$

Let us now suppose that between A and B there are various small inequalities; each of these inequalities will produce small waves whose nature is determined by the form of the functions w, w'; hence w and w' will both be small quantities and the sign of y will be independent of that of $w - w'$. Now f is positive or negative according as the bottom at A is higher or lower than the bottom at B. *Hence if* $V^2 < gD$ *the upper surface of the water rises when the bottom falls, and falls when the bottom rises; and the converse is the case when* $V^2 > gD$.

Theory of Group Velocity.

405. When a group of waves advances into still water, it is observed that the velocity of the group is less than that of the individual waves of which it is composed. This phenomenon was first explained by Prof. Stokes[1], who regarded the group as formed by the superposition of two infinite trains of waves of equal amplitudes and nearly equal wave lengths, advancing in the same direction.

Let the two trains of waves be represented by $\cos k(Vt - x)$ and $\cos k'(V't - x)$; their resultant is equal to

$$\cos k(Vt - x) + \cos k'(V't - x) = 2\cos \tfrac{1}{2}\{(k'V' - kV)t - (k' - k)x\}$$
$$\times \cos \tfrac{1}{2}\{(k'V' + kV)t - (k' + k)x\}.$$

[1] *Smith's Prize Examination*, 1876; and Lord Rayleigh, "On Progressive Waves"; *Proc. Lond. Math. Soc.* vol. IX.

If $k' - k$, $V' - V$ be small, this represents a train of waves whose amplitude varies slowly from one point to another between the limits 0 and 2, forming a series of groups separated from one another by regions comparatively free from disturbance. The position at time t of the middle of the group which was initially at the origin is given by

$$(k'V' - kV)t - (k' - k)x = 0,$$

which shows that the velocity of propagation U of the group is

$$U = (k'V' - kV)/(k' - k).$$

In the limit when the number of waves in each group is indefinitely great we have $k' = k + \delta k$, $V' = V + \delta V$, whence

$$V = \frac{d(kV)}{dk}.$$

406. In the preceding investigation we have supposed that the pressure at the free surface is either constant or zero; we shall now find the forced waves[1] produced by a surface pressure which is equal to

$$C + \left\{\frac{\sqrt{(x^2+b^2)} + b}{x^2+b^2}\right\}^{\frac{1}{2}} \sin \omega t.$$

Let $x + \xi$, $z + \zeta$ be the coordinates at time t of a particle whose initial position is (x, z); also let P denote the time integral of the velocity potential. Then

$$\xi = \frac{d}{dx}\int_0^t \phi dt = \frac{dP}{dx}, \quad \zeta = \frac{dP}{dz}.$$

Since the motion is small,

$$p = C - g(z + \zeta) - \frac{d\phi}{dt},$$

$$= C - gz - g\frac{dP}{dz} - \frac{d^2P}{dt^2} \quad \text{.....................(33)},$$

the density being taken as unity. The equation of continuity is

$$\frac{d^2P}{dx^2} + \frac{d^2P}{dz^2} = 0 \quad \text{.......................(34)},$$

also if η be the elevation

$$\eta = \frac{dP}{dz} \quad \text{.........................(35)}.$$

[1] Sir W. Thomson, *Phil. Mag.* (5) XXIII. p. 113.

A solution of (34) is evidently

$$\psi(t) = (b - z + \iota x)^{-\frac{1}{2}} \exp\{-gt^2/4(b - z + \iota x)\} \text{......(36)}.$$

Changing ι into $-\iota$, adding and dividing by $\sqrt{2}$ we obtain

$$\psi(t) = r^{-1}\{r - z + b)^{\frac{1}{2}} \cos gt^2 x/4r^2 + (r + z - b)^{\frac{1}{2}} \sin gt^2 x/4r^2\}$$
$$\times \exp\{-gt^2(b - z)/4r^2\} \text{...(37)},$$

where $$r^2 = (b - z)^2 + x^2.$$

It is known from the theory of the Conduction of Heat that (36) and therefore (37) is a solution of the equation[1]

$$g\frac{d\psi}{dz} + \frac{d^2\psi}{dt^2} = 0 \text{ ....(38)},$$

whence if

$$\chi(t) = \int_0^t \psi(\tau)\, d\tau,$$

$\chi(t)$ is also a solution of (38). Let us now assume that

$$P = -\int_0^t \chi(t - \tau) \sin \omega\tau\, d\tau = -\int_0^t \chi(\tau) \sin \omega(t - \tau)\, d\tau,$$

then since $$\chi(0) = 0,$$

$$\frac{dP}{dt} = -\int_0^t \chi'(t - \tau) \sin \omega\tau d\tau,$$

$$= -\int_0^t \psi(t - \tau) \sin \omega\tau d\tau,$$

also since

$$\psi(0) = (r - z + b)^{\frac{1}{2}}/r,$$

$$\frac{d^2P}{dt^2} = -r^{-1}(r - z + b)^{\frac{1}{2}} \sin \omega t - \int_0^t \psi'(t - \tau) \sin \omega\tau d\tau,$$

$$= -r^{-1}(r - z + b)^{\frac{1}{2}} \sin \omega t - \int_0^t \chi''(t - \tau) \sin \omega\tau d\tau.$$

We thus obtain

$$g\frac{dP}{dz} + \frac{d^2P}{dt^2} = -r^{-1}(r - z + b)^{\frac{1}{2}} \sin \omega t$$
$$- \int_0^t \sin \omega(t - \tau) \left\{\frac{d^2\chi}{d\tau^2} + g\frac{d\chi}{dz}\right\} d\tau,$$

$$= -r^{-1}(r - z + b)^{\frac{1}{2}} \sin \omega t.$$

[1] This will be proved in Chapter XXIII.

Whence at the surface where $z=0$, we obtain from (33)

$$p = C + \left\{\frac{(b^2+x^2)^{\frac{1}{2}}+b}{x^2+b^2}\right\}^{\frac{1}{2}} \sin \omega t.$$

The velocity potential is

$$\phi = \frac{dP}{dt} = -\int_0^t \sin \omega (t-\tau) \psi(\tau) \, d\tau \quad \text{............(39)}.$$

and the value of ζ is

$$\zeta = \frac{dP}{dz} = -\frac{1}{g}\left\{\frac{d\phi}{dt} + \frac{(r-z+b)^{\frac{1}{2}}}{r} \sin \omega t\right\} \quad \text{.........(40)}.$$

407. Sir W. Thomson has worked out the value of the elevation η on the assumption that $b=0$. This assumption undoubtedly makes the pressure infinite at the origin excepting for values of t which are equal to $2m\pi/\omega$, but as we shall only investigate the value of η at great distances from the origin, the solution we shall obtain will be sufficiently accurate to represent the motion at such points.

Putting $b=0$, $z=0$, we obtain

$$\psi(t) = (2/x)^{\frac{1}{2}} \sin (gt^2/4x + \tfrac{1}{4}\pi).$$

Let $g/4x = k^2$, then the preceding equation becomes

$$\psi(t) = (2/x)^{\frac{1}{2}} \sin (k^2t^2 + \tfrac{1}{4}\pi),$$

whence if

$$\sigma = k\tau,$$

$$\phi = -2(2/g)^{\frac{1}{2}} \int_0^{kt} \sin \omega (t - \sigma/k) \sin (\sigma^2 + \tfrac{1}{4}\pi) \, d\sigma,$$

$$= (2/g)^{\frac{1}{2}} \int_0^{kt} [\cos \{(\sigma - \tfrac{1}{2}\omega/k)^2 - \tfrac{1}{4}\omega^2/k^2 + \omega t + \tfrac{1}{4}\pi\}$$

$$- \cos \{(\sigma + \tfrac{1}{2}\omega/k)^2 - \tfrac{1}{4}\omega^2/k^2 - \omega t + \tfrac{1}{4}\pi\}] \, d\sigma,$$

$$= (2/g)^{\frac{1}{2}} \int_{-\frac{1}{2}\omega/k}^{kt-\frac{1}{2}\omega/k} \cos (\lambda^2 - \tfrac{1}{4}\omega^2 k^2 + \omega t + \tfrac{1}{4}\pi) \, d\lambda,$$

$$- (2/g)^{\frac{1}{2}} \int_{\frac{1}{2}\omega/k}^{kt+\frac{1}{2}\omega/k} \cos (\lambda^2 - \tfrac{1}{4}\omega^2 k^2 - \omega t + \tfrac{1}{4}\pi) \, d\lambda \quad \text{.................(41)}.$$

Let x be very large, and let t be so large that $kt - \frac{1}{2}\omega/k$ is a large positive quantity. Then k is small and the second integral vanishes, whilst the limits of the first are ∞ and $-\infty$, whence remembering that

$$\int_0^\infty (\sin \textit{or} \cos) \, \lambda^2 d\lambda = (\tfrac{1}{8}\pi)^{\frac{1}{2}},$$

we obtain

$$\phi = (2\pi/g)^{\frac{1}{2}} \cos(\omega^2 x/g - \omega t),$$

and

$$\eta = \omega(2\pi/g^3)^{\frac{1}{2}} \{\sin(\omega^2 x/g - \omega t) - (g/2\pi\omega^2 x)^{\frac{1}{2}} \sin \omega t\},$$

$$= \omega(2\pi/g^3)^{\frac{1}{2}} \sin(\omega^2 x/g - \omega t) \quad \text{.........................(42)},$$

approximately, since the first term is large compared with the second.

Hence

$$\lambda = 2\pi g/\omega^2, \quad U = g/\omega = (g\lambda/2\pi)^{\frac{1}{2}}.$$

We therefore see that at great distances from the origin, where the pressure is approximately constant, the waves are approximately the same as a procession of *free* waves.

On the other hand if x is large and t so small that $kt - \frac{1}{2}\omega/k$ is a large negative quantity, both integrals vanish; and wave motion does not exist. Hence as the time advances wave motion gradually commences from nothing until it becomes the regular procession of waves represented by (42) and so continues for ever afterwards.

When x is large, the value of ϕ at the time $t = 2\omega x/g$, is

$$\phi = (2/g)^{\frac{1}{2}} \int_0^\infty \cos(\lambda^2 - \tfrac{1}{4}\omega^2 k^2 + \omega t + \tfrac{1}{4}\pi)\, d\lambda,$$

$$= (\pi/2g)^{\frac{1}{2}} \cos(\omega^2 x/g - \omega t),$$

and therefore ϕ has attained half its final value. The point x where this condition is fulfilled at time t may be called the mid-front of the procession. It travels with the velocity $\frac{1}{2}g/\omega$ or half the wave velocity.

Deep Sea Waves.

408. In § 387 we determined the motion of deep sea waves upon the assumption that the motion is slow enough to allow the squares and products of the velocities to be neglected. A higher degree of accuracy might be obtained by substituting the solution we have already obtained in the terms of the second order, and proceeding by the usual method of successive approximation. This mode of proceeding is however somewhat laborious, and we shall therefore employ a different method which is due to Prof. Stokes[1].

[1] *Math. and Phys. Papers*, vol. I. p. 314.

Since ϕ and ψ are conjugate functions of x and z, we have

$$dx/d\phi = dz/d\psi, \text{ and } dz/d\psi = -dz/d\phi;$$

whence if

$$S = \left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dx}{d\psi}\right)^2 = \left(\frac{dz}{d\phi}\right)^2 + \left(\frac{dz}{d\psi}\right)^2,$$

and if we change the independent variables from x and z to ϕ and ψ, we obtain

$$Sd\phi/dx = dz/d\psi, \quad Sd\phi/dz = -dx/d\psi,$$

whence $$u^2 + w^2 = S^{-1},$$

and $$p/\rho + g(z - C/m) + (2S)^{-1} = 0,$$

where C and m are constants.

Let us convert the wave motion into steady motion by impressing on the whole liquid a velocity $-U$, where U is the velocity of propagation of the waves. If there were no wave motion we should have $\phi = -Ux$, whence we may assume

$$x = -\phi/U + m^{-1}\Sigma_1^\infty (B_r\epsilon^{rm\psi/U} + A_r\epsilon^{-rm\psi/U})(\sin \textit{ or } \cos)\, rm\phi/U,$$

$$z = -\psi/U + m^{-1}\Sigma_1^\infty (B_r\epsilon^{rm\psi/U} - A_r\epsilon^{-rm\psi/U})(\cos \textit{ or } \sin)\, rm\phi/U,$$

where r is a positive integer. If λ be the wave length, the value of x when ϕ is changed into $\phi - 2\pi U/m$ must be $x + \lambda$; whence $m = 2\pi/\lambda$. Also if $\psi = 0$ be the equation of the free surface and the origin of x and ϕ be taken in the trough of the wave, z must be a maximum when $\phi = \pi U/m$; whence the cosine terms in x, and the sine terms in z must disappear. Since z is measured upwards and $Uz = -\psi$ in the undisturbed motion, ψ must increase with the depth of the liquid, whence the B's vanish. If therefore we write for shortness ϕ and ψ for $m\phi/U$ and $m\psi/U$, the values of x and z may finally be written

$$\left.\begin{aligned} x &= -\phi/m + m^{-1}\Sigma_1^\infty A_r\epsilon^{-r\psi}\sin r\phi \\ z &= -\psi/m - m^{-1}\Sigma_1^\infty A_r\epsilon^{-r\psi}\cos r\phi \end{aligned}\right\}\ldots\ldots\ldots\ldots(43),$$

where the A's have to be determined. At the free surface p and ψ are zero, whence

$$(z - C/m)S + (2g)^{-1} = 0.$$

Substituting the values of z and S obtained from (43), we find

$$(C + \Sigma A_r \cos r\phi)[1 - 2\Sigma r A_r \cos r\phi + \Sigma r^2 A_r^2 + 2\Sigma rs A_r A_s \cos(r-s)\phi] - U^2 m/2g = 0\ldots(44),$$

where in the term in the square brackets, each different combination of the letters r and s is to be taken once.

This equation may be arranged in the form

$$B_0 + B_1 \cos\phi + B_2 \cos 2\phi + \ldots = 0,$$

and since it has to be satisfied independently of ϕ, we must have

$$B_0 = 0,\ \ B_1 = 0,\ \ B_2 = 0 \ \&\text{c.} \qquad\qquad (45).$$

Let $A_1 = b$; then we shall make the assumption which will be justified by the result, that A_r is a quantity of the order b^r, and we shall endeavour to obtain an approximate solution as far as the terms involving b^5. Equations (45) written out at full length as far as the terms of the order b^5 become,

$$\left.\begin{aligned}
C(1 + A_1^2 + 4A_2^2) - A_1^2 + 2A_1^2A_2 - 2A_2^2 - U^2m/2g &= 0\\
C(-2A_1 + 4A_1A_2 + 12A_2A_3) + A_1 + A_1^3 - 3A_1A_2 &\\
+ 6A_1A_2^2 + 3A_1^2A_3 - 5A_2A_3 &= 0\\
C(-4A_2 + 6A_1A_3) + A_2 - A_1^2 + 3A_1^2A_2 - 4A_1A_3 &= 0\\
C(-6A_3 + 8A_1A_4) + A_3 - 3A_1A_2 + 4A_1^2A_3 + 2A_1A_2^2 &\\
- 5A_1A_4 &= 0\\
-8CA_4 + A_4 - 4A_1A_3 - 2A_2^2 &= 0\\
-10CA_5 + A_5 - 5A_1A_4 - 5A_2A_3 &= 0
\end{aligned}\right\}\ (46).$$

In order to obtain a first approximation, we must reject all the terms except those of the lowest order in each equation, and we shall obtain

$$C = U^2m/2g,\ \ C = \tfrac{1}{2},\ \ A_2 = -b^2,\ \ A_3 = \tfrac{3}{2}b^3,\ \ A_4 = -\tfrac{8}{3}b^4,\ \ A_5 = \tfrac{125}{24}b^5,$$

whence $U^2 = g/m = g\lambda/2\pi$ as before.

Let us now put

$$C = \tfrac{1}{2} + x,\ \ A_2 = -b^2 - y,\ \ A_3 = \tfrac{3}{2}b^3 + z,$$

where x, y, z are at least of the orders b, b^3, b^4 respectively. Substituting in the second, third and fourth of (46), and retaining terms of one order higher, we shall obtain

$$x = b^2,\ \ y = \tfrac{1}{2}b^4,\ \ z = \tfrac{19}{12}b^5,$$

whence $$-A_2 = b^2 + \tfrac{1}{2}b^4,\ \ A_3 = \tfrac{3}{2}b^3 + \tfrac{19}{12}b^5.$$

Lastly substituting these values of A_2, A_3 in the second of (46) we obtain

$$C = \tfrac{1}{2} + b^2 + \tfrac{11}{4}b^4,$$

and hence the final equations are

$$U^2 = gm^{-1}(1 + b^2 + \tfrac{7}{2}b^4),$$

$$mx = -\phi + b\epsilon^{-\psi}\sin\phi - (b^2 + \tfrac{1}{2}b^4)\epsilon^{-2\psi}\sin 2\phi + (\tfrac{3}{2}b^3 + \tfrac{19}{12}b^5)\epsilon^{-3\psi}\sin 3\phi$$
$$- \tfrac{8}{3}b^4\epsilon^{-4\psi}\sin 4\phi + \tfrac{125}{24}b^5\epsilon^{-5\psi}\sin 5\phi,$$

$$mz = -\psi - b\epsilon^{-\psi}\cos\phi + (b^2 + \tfrac{1}{2}b^4)\epsilon^{-2\psi}\cos 2\phi - (\tfrac{3}{2}b^3 + \tfrac{19}{12}b^5)\epsilon^{-3\psi}\cos 3\phi$$
$$+ \tfrac{8}{3}b^4\epsilon^{-4\psi}\cos 4\phi - \tfrac{125}{24}b^5\epsilon^{-5\psi}\cos 5\phi.$$

In order to obtain the equation of the free surface, we must put $\psi = 0$ in the preceding equations, and we find

$$mx = -\phi + b\sin\phi - (b^2 + \tfrac{1}{2}b^4)\sin 2\phi + (\tfrac{3}{2}b^3 + \tfrac{19}{12}b^5)\sin 3\phi$$
$$- \tfrac{8}{3}b^4\sin 4\phi + \tfrac{125}{24}b^5\sin 5\phi \ldots (47),$$

$$mz = -b\cos\phi + (b^2 + \tfrac{1}{2}b^4)\cos 2\phi - (\tfrac{3}{2}b^3 + \tfrac{19}{12}b^5)\cos 3\phi$$
$$+ \tfrac{8}{3}b^4\cos 4\phi - \tfrac{125}{24}b^5\cos 5\phi \ldots (48),$$

and the equation of the wave profile is determined by eliminating ϕ between (47) and (48).

The elimination is most easily effected by Lagrange's theorem, and gives

$$-mz + \tfrac{1}{2}b^2 + b^4 = (b + \tfrac{9}{8}b^3)\cos mx - (\tfrac{1}{2}b^2 + \tfrac{11}{6}b^4)\cos 2mx + \tfrac{8}{3}b^3\cos 3mx$$
$$- \tfrac{1}{3}b^4\cos 4mx,$$

to the fourth order. Let $b + \frac{9}{8}b^3 = a$, then to the fourth order

$$b = a - \tfrac{9}{8}a^3,$$

and shifting the origin so as to get rid of the constant term, the equation of the wave profile may finally be written

$$mz = -a\cos mx + (\tfrac{1}{2}a^2 + \tfrac{17}{24}a^4)\cos 2mx - \tfrac{3}{8}a^3\cos 3mx + \tfrac{1}{3}a^4\cos 4mx.$$

Now the equations of a trochoid are given by the equations

$$mx = \alpha\theta + \beta\sin\theta, \quad -mz = \beta\cos\theta + \gamma.$$

In order that x may have the same period in the trochoid as in the wave profile, we must have $\alpha = 1$. We then obtain by development of the fourth order, and choosing γ so as to make the constant term vanish

$$-mz = (\beta - \tfrac{3}{8}\beta^3)\cos mx - (\tfrac{1}{2}\beta^2 - \tfrac{1}{3}\beta^4)\cos 2mx + \tfrac{3}{8}\beta^3\cos 3mx$$
$$- \tfrac{1}{3}\beta^4\cos 4mx,$$

and putting $\beta - \frac{3}{8}\beta^3 = a$, we obtain to the fourth order

$$mz = -a\cos mx + (\tfrac{1}{2}a^2 + \tfrac{1}{24}a^4)\cos 2mx - \tfrac{3}{8}a^3\cos 3mx + \tfrac{1}{3}a^4\cos 4mx.$$

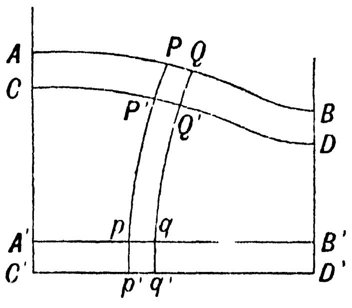
Hence if z_w, z_t denote the ordinates of the wave and trochoid respectively

$$z_w - z_t = \tfrac{2}{3}a^4m^{-1}\cos 2mx.$$

Hence to the third order the form of the wave profile is a trochoid, but if we proceed to the fourth order we see that the wave lies a little above the trochoid at the crest and trough, and a little below it at the shoulders.

Prof. Stokes has also applied the same method to investigate the form of the waves propagated in a liquid of finite depth, but the results are naturally more complicated, and we must therefore refer the reader to his Collected Papers[1].

409. Professor Stokes has also shown[2] that in addition to the wave motion, the liquid has a slow motion of translation in the direction of the wave, which rapidly diminishes with the depth of the liquid. Lord Rayleigh[3] has given an elegant geometrical proof that this motion is a consequence of the absence of molecular rotation, and is independent of the condition of constant pressure at the free surface.



Let AB be the surface from crest to hollow, and CD a neighbouring stream line. Let us suppose the motion is made steady by reversing the velocity of propagation, and draw two stream lines $A'B'$, $C'D'$ at such a depth that the steady motion of the liquid is uniform, and so that the flow across $A'C'$ is equal to the flow across AC. Then we have to show that a particle at A will take longer to reach B, than a particle at A' takes to arrive at B'. Now if σ denotes the small breadth of the tube of flow AD, and V the velocity, the total stream is σv and is constant and

[1] Vol. I. p. 320.

[2] *Math. and Phys. Papers*, vol. I. p. 207.

[3] *Phil. Mag.* April, 1876.

equal to K suppose. The time occupied by a particle in travelling from A to B is therefore

$$t = \int v^{-1}\, ds = K^{-1} \int \sigma ds = K^{-1} \text{ area } AD.$$

Hence if t' is the time from A' to B',

$$t' = K^{-1} \text{ area } A'D',$$

and since K is the same in both cases,

$$t : t' :: \text{area } AD : \text{area } A'D',$$

and it remains to show that area $AD >$ area $A'D'$.

Let us draw a series of equipotential lines ϕ and $\phi + d\phi$, such that the small spaces between them and AB, CD are squares.

Then $PQ = d\phi/v$, $PP' = d\psi/v$ and therefore $d\phi = d\psi$: also $pq = d\phi/v'$, $pp' = d\psi'/v'$, but since the flux across AC, and $A'C'$ are the same, $d\psi' = d\psi = d\phi$, whence $pq = pp'$ and therefore the equipotential lines divide $A'D'$ into squares. Now if a line be divided into a given number of parts, the sum of the squares of all the parts will be a minimum when the parts are all equal[1]. Hence the space AD is greater than if the squares described on AB were all equal, and therefore a fortiori greater than the space $A'D'$ which consists of the sum of the squares of equal parts of a shorter line.

Hence it follows that when a particle starting from A' has arrived at B', another particle starting at the same moment from A will fall short of B. Thus in a progressive wave, the water near the surface has on the whole a motion of translation in the direction in which the waves advance.

[1] This may be proved as follows. Let

$$U^2 = x^2 + y^2 + u^2, \qquad \lambda = x + y + \mu,$$

where λ is the length of the line; x and y the lengths of any two parts; u^2 and μ the sum of the squares, and the sum of the remaining parts respectively, then

$$\begin{aligned} \lambda^2 - U^2 &= 2xy + 2\mu(x+y) - u^2 \\ &= \tfrac{1}{2}(x+y)^2 - \tfrac{1}{2}(x-y)^2 + 2\mu(x+y) - u^2. \end{aligned}$$

Hence $\lambda^2 - U$ will be a maximum, and therefore U^2 will be a minimum when $x = y$.

SECTION II.

The Solitary Wave.

410. The theory of irrotational waves of permanent type depends upon the assumption, that it is possible for an infinite train of similar waves to follow one another without suffering degradation of form. The experiments described by the late Mr Scott Russell[1] indicate that it is possible for a single wave to be propagated along the surface of a liquid, and such a wave is called by him a solitary wave. He states that the length of the wave is about six or eight times the depth of the liquid, and therefore partakes of the character of a long wave; but that it possesses several peculiarities, the principal of which are that a positive wave or elevation is capable of being propagated to a considerable distance without breaking up, whilst a negative wave or depression is incapable of being propagated to any considerable distance without becoming dissipated.

The mathematical theory of the solitary wave has been in former times the subject of considerable controversy; it was discussed by Earnshaw[2] in 1845, but his theory has not been regarded as satisfactory. A satisfactory approximate theory was given by Boussinesq[3] in 1871, and a very similar one was discovered independently by Lord Rayleigh[4] in 1876. We shall now proceed to consider the theory of the latter.

411. We shall suppose that the motion is in two dimensions, and that the bottom of the liquid is horizontal. Let the origin be taken in the bottom of the liquid, and let the axis of x be measured in the direction of propagation of the wave, whilst the axis of y is measured vertically upwards. Let l be the depth of the liquid when undisturbed, l' the height of the crest above the bottom of the liquid.

[1] *Brit. Assoc. Rep. on Waves*, 1844.

[2] *Trans. Camb. Phil. Soc.* vol. VIII. p. 326.

[3] *Comptes Rendus*, vol. LXII.

[4] *Phil. Mag.* Ap. 1876; See also Airy, *Tides and Waves;* Stokes, *Brit. Assoc. Rep. on Hydrodynamics*, 1845.

Since the motion is irrotational, the current function ψ satisfies Laplace's equation, and we may therefore put

$$\psi = \sin\left(y\frac{d}{dx}\right)F(x) = yf(x) - \frac{y^3}{3\,!}f''(x) + \frac{y^5}{5\,!}f^{\text{IV}}(x) - \ldots (49)$$

where $f(x) = F'(x)$. Since the motion is steady, the pressure is determined by the equation

$$p/\rho + gy + \tfrac{1}{2}(u^2 + v^2) = A.$$

Putting $A - p/\rho = \frac{1}{2}\varpi$, this becomes

$$u^2 + v^2 = \varpi - 2gy \ldots\ldots\ldots\ldots\ldots\ldots (50).$$

At the free surface ϖ must be constant; if therefore we can determine y as a function of x, such that ϖ shall be constant at the free surface, this relation will determine its form.

Since $u^2 + v^2 = (1 + y'^2)u^2$ where $y' = dy/dx$, (50) may be written

$$yu = (\varpi y^2 - 2gy^3)^{\frac{1}{2}}/(1 + y'^2)^{\frac{1}{2}}.$$

Now

$$yu = y\frac{d\psi}{dy} = yf - \frac{y^3}{2\,!}f'' + \frac{y^5}{4\,!}f^{\text{IV}} - \ldots\ldots = \sqrt{\frac{\varpi y^2 - 2gy^3}{1 + y'^2}} \ldots (51).$$

The function f is the value of u at the bottom of the liquid and is very nearly constant, and therefore $f(x)$ varies very slowly; hence the differential coefficients of $f(x)$ are small quantities. Also if the curvature of the wave profile is small, y', y'' ... will also be small quantities, and we may therefore eliminate f between (49) and (51) by successive approximation. Since ψ is constant at the free surface, we have writing $R = \psi/y$,

$$f = R,\quad f'' = R'';$$

whence to the second order

$$\begin{aligned} f &= R + \tfrac{1}{6}y^2R'' - \tfrac{1}{120}y^4R^{\text{IV}} + \\ f'' &= R'' + \tfrac{1}{6}y^2R^{\text{IV}} - \tfrac{1}{3}(y''y + y'^2)R'' + \tfrac{2}{3}y''yR''', \\ f^{\text{IV}} &= R^{\text{IV}} + \&\text{c}. \end{aligned}$$

neglecting terms of the fifth order. Substituting in (51) we obtain

$$\psi\left[1 - \tfrac{1}{3}y^3\left(\frac{1}{y}\right)'' - \tfrac{1}{45}y^5\left(\frac{1}{y}\right)^{\text{IV}} + \tfrac{1}{6}y^3\left\{(y''y + y'^2)\left(\frac{1}{y}\right)'' - 2y'y\left(\frac{1}{y}\right)'''\right\} + \ldots\right]$$
$$= \sqrt{\frac{\varpi y^2 - 2gy^3}{1 + y'^2}}.$$

If we neglect terms of the fourth and higher orders, this equation becomes,

$$\psi^2 (1 + y'^2) \{1 + \tfrac{1}{3} (yy'' - 2y'^2)\}^2 = \varpi y^2 - 2gy^3,$$

or

$$\psi^2 (1 - \tfrac{1}{3} y'^2 + \tfrac{2}{3} yy'') = \varpi y^2 - 2gy^3.$$

The above equation may be put into the form

$$\frac{d^2 y^{\frac{1}{2}}}{dx^2} = \tfrac{3}{4} \psi^{-2} (\varpi y^{\frac{1}{2}} - 2gy^{\frac{3}{2}} - \psi^2 y^{-\frac{3}{2}}).$$

Multiplying by $2dy^{\frac{1}{2}}dx$ and integrating we obtain

$$\tfrac{1}{3} (dy/dx)^2 = Cy + (\varpi y^2 - gy^3)/\psi^2 + 1 \ldots\ldots\ldots\ldots (52).$$

Let u_0 be the velocity of the liquid in the undisturbed parts of the stream, then

$$\varpi = u_0^2 + 2gl \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53),$$

and

$$\psi = \int_0^l u_0 dy = u_0 l \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54).$$

whence (52) becomes

$$\tfrac{1}{3} (dy/dx)^2 = 1 + Cy + y^2 (u_0^2 + 2gl)/u_0^2 l^2 - gy^3/u_0^2 l^2 \ldots\ldots (55).$$

In this equation g and l are given, whilst C and u_0 are at our disposal; hence the cubic expression on the right hand side of (55) may be made to vanish when $y = l$ and $y = l'$. If we substitute these values of y and equate the right hand side of (55) to zero, we shall obtain

$$u_0^2 = gl' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56),$$

$$-Cl = 2 + gl/u_0^2 = 2 + l/l'.$$

Substituting these values of u_0 and C, (55) becomes,

$$(dy/dx)^2 + 3 (y - l)^2 (y - l')/l^2 l' = 0 \ldots\ldots\ldots\ldots (57).$$

From this equation it appears that there is only one maximum or minimum value of y besides l; and since $y - l'$ is necessarily negative, the surface condition cannot be satisfied to this order of approximation by a solitary wave of depression.

Differentiating (59) we obtain

$$d^2y/dx^2 = \tfrac{3}{2} (y - l) (2l' + l - 3y)/l^2 l',$$

which shows that the points of zero curvature occur when $y = l$ and $y = \tfrac{1}{3} (2l' + l) = l + \tfrac{2}{3} (l' - l)$. Hence the curvature changes

sign at two-thirds of the height of the wave above the undisturbed level, and at this point only.

If we put $l' - l = \beta$, $y - l = \eta$ and integrate (57), we shall obtain

$$\eta = \beta \operatorname{sech} \tfrac{1}{2}x\,(3\beta/l^2 l')^{\frac{1}{2}},$$

the constant being chosen so that $x = 0$ when $\eta = \beta$. This equation determines the form of the wave profile, and it therefore follows that when the depth of the liquid and the velocity of propagation are given, there is only one solitary wave. On either side of the greatest elevation the height diminishes indefinitely, but does not absolutely vanish; hence there is no definite wave length.

If we regard the wave as ending where the height is one tenth of the maximum, we obtain

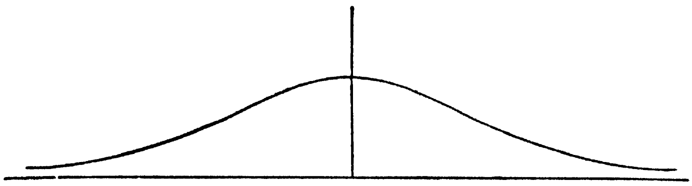
$$x/l = 2 \cdot 1\sqrt{(1 + l/\beta)}.$$

The shortest wave length is when $\beta = l$ and then

$$2x/l = 5{\cdot}96.$$

If $\beta = \frac{1}{3}l$; $2x/l = 8{\cdot}4$; and if $\beta = \frac{1}{8}l$, $2x/l = 12{\cdot}6$. These results agree with Scott Russell's observations.

The form of the wave is shown in the figure, and its velocity of



propagation is given by (56), which is the value deduced by Scott Russell from his experiments.

Another of Scott Russell's observations is now readily accounted for:—He found that the wave broke when its elevation above the general level became equal, or nearly so, to the depth of the undisturbed liquid. If V be the velocity of the liquid at the crest of the wave we obtain from (50)

$$\begin{aligned} V^2 &= \varpi - 2gl', \\ &= g\,(2l - l'), \end{aligned}$$

by (53) and (55); which requires that $l > l' - l$. When therefore the wave is on the point of breaking, the water at the crest is moving with the velocity of the wave.

SECTION III.

Capillary Waves.

412. We must now consider the third class of waves, which are principally due to capillary forces.

Let T be the surface tension of the liquid; δp the excess of pressure in the liquid just below the free surface; then

$$\delta p/\rho + g\eta + \dot{\phi} = 0.$$

But if r, r' denote the radii of curvature of two vertical sections in and perpendicular to the direction of propagation of the waves

$$-\delta p = T(r^{-1} + r'^{-1}) = T\left(\frac{d^2\eta}{dx^2} + \frac{d^2\eta}{dy^2}\right),$$

since the curvature is supposed to be small at the free surface; whence

$$T\left(\frac{d^2\eta}{dx^2} + \frac{d^2\eta}{dy^2}\right) = g\rho\eta + \rho\dot{\phi}.$$

Differentiating with respect to t, and remembering that $\dot{\eta} = d\phi/dz$, and that $\nabla^2\phi = 0$, the above equation becomes[1]

$$-T\frac{d^3\phi}{dz^3} = g\rho\frac{d\phi}{dz} - g\rho l^{-1}\phi,$$

or

$$l\frac{d\phi}{dz} + \frac{Tl}{g\rho}\frac{d^3\phi}{dz^3} = \phi \quad \text{.....................(58)},$$

where l is the length of the simple equivalent pendulum.

413. We shall now apply the preceding result to determine the capillary waves propagated along a canal of depth h.

Assuming as usual that

$$\phi = A\cosh m(z+h)\cos(mx - nt),$$

and substituting in (58), we obtain

$$ml\sinh mh + Tlm^3g^{-1}\rho^{-1}\sinh mh = \cosh mh.$$

Whence $\quad U^2 = n^2/m^2 = g(m^{-1} + Tm/g\rho)\tanh mh,$

$$= (g\lambda/2\pi + 2\pi T/\rho\lambda)\tanh 2\pi h/\lambda \quad \text{......(59)}.$$

Equation (59) determines the wave length corresponding to a given velocity of propagation.

[1] Kolacek, *Fortschritte der Mathematic*, 1878.

Let us now suppose that the depth of the liquid is so great that tanh $2\pi h/\lambda$ may be replaced by unity. Equation (59) becomes

$$g\rho\lambda^2 - 2\pi\rho U^2\lambda + 4\pi^2 T = 0 \quad\quad (60),$$

whence $$\lambda = \pi U^2/g \pm \pi g^{-1} \sqrt{(U^4 - 4Tg/\rho)}.$$

In order that wave motion may be possible both values of λ must be real, which requires that

$$U = \text{or} > (4Tg/\rho)^{\frac{1}{4}}.$$

Hence the minimum value of U is $(4Tg/\rho)^{\frac{1}{4}}$, and the corresponding value of λ is $2\pi\sqrt{(T/g\rho)}$.

Sir W. Thomson defines a *ripple* to be a wave whose length is less than the preceding critical value of λ[1].

414. In § 389 we have considered the propagation of waves at the surface of separation of two liquids which are moving with different velocities. We shall now consider the production of ripples by wind blowing over the surface of still water.

Let V be the velocity of the wind, which is supposed to be parallel to the undisturbed surface of the water, σ the density of air referred to water.

Since the changes of density of the air are very small in the neighbourhood of the water, the air may approximately be regarded as an incompressible fluid, whence if the accented letters refer to the water, the kinematical conditions at the boundary give

$$\phi = Vx + a(U - V)\,\epsilon^{-mz}\cos(mx - nt),$$

$$\phi' = -aU\epsilon^{mz}\cos(mx - nt),$$

where U is the velocity of propagation of the waves in the water, and $\eta = a\sin(mx - nt)$ is the equation of its free surface.

The dynamical condition at the free surface is

$$\delta p' - \delta p = T\frac{d^2\eta}{dx^2} \quad\quad (61).$$

Now

$$\delta p/\sigma + g\eta + \dot{\phi} + \tfrac{1}{2}\{V - am(U - V)\sin(mx - nt)\}^2 - \tfrac{1}{2}V^2 = 0,$$

or $$\delta p + a\sigma\{g + n(U - V) - mV(U - V)\}\sin(mx - nt) = 0.$$

Similarly

$$\delta p' + (g - Un)\,a\sin(mx - nt) = 0,$$

[1] *Phil. Mag.* (4), vol. XLII.

whence (61) becomes

$$g(\sigma - 1) + \sigma m (U - V)^2 + m U^2 - T m^2 = 0 \quad \ldots\ldots(62)$$

Let W be the velocity of propagation of waves in water when there is no wind, then

$$W = \sqrt{\frac{g(1-\sigma) + Tm^2}{m(1+\sigma)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(63),$$

or

$$Tm^2 - m(1+\sigma) W^2 + g(1-\sigma) = 0.$$

The condition that the roots of this quadratic in m should be real is that

$$W^2 = \text{or} > \frac{2}{1+\sigma} \sqrt{Tg(1-\sigma)} \ldots\ldots\ldots\ldots\ldots(64),$$

which determines the minimum value of W. This value of W is less than $(4Tg)^{\frac{1}{4}}$, which shows that when water is in contact with air, it is possible for ripples to travel over its surface.

Substituting the value of W from (63) in (62) we obtain

$$(1+\sigma) U^2 - 2\sigma V U + \sigma V^2 - (1+\sigma) W^2 = 0,$$

whence

$$U = \frac{\sigma V}{1+\sigma} \pm \sqrt{\left\{ W^2 - \frac{\sigma V^2}{(1+\sigma)^2} \right\}} \quad \ldots\ldots\ldots\ldots(65).$$

We shall now discuss this equation.

Case (i). $V < W\sqrt{(1+\sigma)/\sigma}$.

In this case both values of U are real, and one of them is positive and the other negative; hence waves can travel either with or against the wind. Moreover since the positive value is numerically greater than the negative value, waves travel faster with the wind, than against the wind; also the velocity of waves travelling against the wind is always less than W.

Case (ii). $V > W\sqrt{(1+\sigma)/\sigma}$.

In this case both values of U if real, are positive; hence waves cannot travel against the wind.

Case (iii). When $V < 2W$, the velocity of waves travelling with the wind is $> W$; when $V > 2W$ this velocity is $< W$; and when $V = 2W$, the velocity of waves travelling with the wind is undisturbed.

Case (iv). If $V > W(1+\sigma)\sigma^{-\frac{1}{2}}$, both values of U are imaginary which shows that the motion is unstable.

Waves in Ice of Uniform Thickness Resting on Water.

415. If the upper surface of water be covered with ice and if any disturbance be given to the water, the elasticity of the ice will cause waves consisting of *lateral* vibrations to be propagated along it.

Let L be the flexural rigidity of ice, σ the mass of a section of unit of area, the equation of motion of the ice is

$$\sigma\ddot{\zeta} = -L\frac{d^4\zeta}{dx^4} + \delta p \quad \text{.....................(66).}$$

Let E be Young's modulus of elasticity, e the thickness of the ice, then neglecting the slight difference between the density of water and ice, we have

$$L = \tfrac{1}{12}e^3E, \quad \sigma = e\rho.$$

Let the velocity potential of the water be

$$\phi = A\cosh m(z+h)\cos(mx-nt),$$

then
$$\zeta = -Amn^{-1}\sinh mh\sin(mx-nt),$$

and
$$\delta p + g\rho\zeta + \rho An\cosh mh\sin(mx-nt) = 0.$$

Substituting in (66) we obtain

$$(e + m^{-1}\coth mh)\,U^2 = \tfrac{1}{12}e^3m^2E + g\rho/m^2,$$

or
$$U^2 = \frac{\frac{2}{3}\pi^3e^3E/\lambda^3\rho + g\lambda/2\pi}{2\pi e/\lambda + \coth(2\pi h/\lambda)}.$$

It may be stated that ice was the first substance for which an experimental determination of E was attempted (see Young's *Lectures on Natural Philosophy*).

Further examples of waves in water covered with ice will be found in Prof. Greenhill's Article on Waves.

In addition to the papers referred to in the text, the reader may consult the following authorities.

Cauchy, *Mém. des Savants étrangers*, vol. I. 1827.

Poisson, *Mém. de l'Institute*, vol. I. 1816.

Green, *Trans. Camb. Phil. Soc.* 1838.

Kelland, *Trans. Roy. Soc. Edin.* vols. XIV. and XV.

Lord Rayleigh, "On Progressive Waves," *Proc. Lond. Math. Soc.* vol. IX.

Lord Rayleigh, "The Form of Standing Waves on the Surface of Running Water," *Ibid.* vol. XV.

Lord Rayleigh, "On the Vibrations of a Cylindrical Vessel containing Liquid," *Phil. Mag.* June, 1883.

Sir W. Thomson, "On Stationary Waves in Flowing Water," *Phil. Mag.* (5), vol. XXII. pp. 353, 445, 517 ; and vol. XXIII. p. 52.

Sir W. Thomson, "On the Front and Rear of a Free Procession of Waves," *Ibid.* vol. XXIII. p. 113.

Sir W. Thomson, "On the Waves produced by a Single Impulse in Water of any Depth," *Ibid.* p. 252.

Greenhill, "On Wave Motion in Hydrodynamics," *American Journal of Mathematics*, vol. IX.

An account of the principal memoirs on wave motion is given by Saint-Venant, in an article, "De la Houle et du Clapotis," *Annales des Ponts et Chaussées*, May, 1888.

EXAMPLES.

1. A liquid of infinite depth is bounded by a fixed plane perpendicular to the direction of propagation of the waves. Prove that each element of liquid will vibrate in a straight line, and draw a figure representing the free surface and the direction of motion of the elements, when the crest of the wave reaches the fixed plane.

2. Prove that the velocity of propagation of long waves in a semi-circular canal of radius a and whose banks are vertical, is

$$\tfrac{1}{2}(\pi g a)^{\frac{1}{2}}.$$

3. If two series of waves of equal amplitude and nearly equal wave length travel in the same direction, so as to form alternate lulls and roughness, prove that in deep water these are propagated with half the velocity of the waves; and that as the ratio of the depth to the wave length decreases from ∞ to 0, the ratio of the two velocities of propagation increases from $\frac{1}{2}$ to 1.

4. If a small system of rectilinear waves move parallel to and over another large rectilinear system, prove that the path of a particle of water is an epicycloid or hypocycloid, according as the two systems are moving in the same or opposite directions.

5. If a cylinder is bounded by $r=a$, and $\theta=0$, $\theta=\frac{2}{3}\pi$, prove that if n is the least number of oscillations per second in a liquid of depth h in the cylinder,

$$\phi = A\,(kr)^{-\frac{1}{2}}\{(kr)^{-1}\sin kr - \cos kr\}\cos\tfrac{3}{2}\theta\cosh kz\cos 2\pi nt$$

where $\qquad (3-2k^2a^2)\tan ka = 3ka, \quad n^2 = gk\tanh kh/4\pi^2.$

6. A fine tube made of a thin slightly elastic substance is filled with liquid; prove that the velocity of propagation of a disturbance in the liquid is $(\lambda\theta/a\rho)^{\frac{1}{2}}$, where a is the internal diameter of the tube, θ its thickness, λ the coefficient of elasticity of the material of which it is made, and ρ the density of the liquid.

7. A circular canal of radius a and of breadth very small compared with a, has its sides vertical and contains liquid of depth d. An isosceles right-angled prism whose length is equal to the breadth of the canal, floats immersed to a depth b in the liquid with its parallel edges coinciding with the radii of the canal, and its hypothenuse horizontal. If the prism be suddenly removed without disturbing the liquid, show that the velocity potential of the resulting motion is

$$gbt/2\pi a + 2\,(2g)^{\frac{1}{2}}a^{\frac{3}{2}}/\pi n^{\frac{5}{2}}\,.\,\Sigma_1^{\infty}\sin^2 nb/2a\,.\,(\sin 2nd/a)^{-\frac{1}{2}}$$
$$\times\cosh n\,(z+d)/a\,.\,\cos n\theta\sin\,(gna^{-1}\tanh nd/a)^{\frac{1}{2}}\,t.$$

8. A horizontal rectangular box is completely filled with three liquids which do not mix, whose densities reckoned downwards are σ_1, σ_2, σ_3, and whose depths when in equilibrium are l_1, l_2, l_3 respectively. Show that if long waves are propagated at their common surfaces, the velocity of propagation V must satisfy the equation

$$\{(\sigma_1/l_1+\sigma_2/l_2)\,V^2-g\,(\sigma_2-\sigma_1)\}\,\{(\sigma_2/l_2+\sigma_3/l_3)\,V^2-g\,(\sigma_3-\sigma_2)\}=\sigma_2^{\,2}\,V^4/l_2^{\,2}.$$

9. A given mass of air is at rest in a circular cylinder of radius c under the action of a constant force to the axis; show that if the force suddenly cease to act, the velocity potential at any subsequent time varies as

$$\Sigma\,\frac{J_0\,(kr)}{k^2J_0\,(kc)}\sin kat.$$

where a is the velocity of sound in air, and the summation extends to all values of k satisfying $J_1\,(kc)=0$, and the square of the condensation is neglected.

10. Prove that liquid of density ρ flowing with mean velocity U through an elastic tube of radius a, will throw the surface into slight stationary corrugations, of which the number per unit of length is

$$(2\rho a\,U^2-\lambda)^{\frac{1}{2}}/(2\pi aT)^{\frac{1}{2}},$$

where λ is the modulus of elasticity of the substance of the tube, and T its total tension.

11. The radius of a solid sphere surrounded by an unlimited mass of air is given by $R(1 + \alpha \sin nat)$, where a is the velocity of sound in air. Show that the mean energy per unit of mass of air at a distance r from the centre of the sphere due to the motion of the latter is

$$\tfrac{1}{4} n^2 a^2 \alpha^2 R^6 (1 + 2n^2 r^2)/r^4 (1 + n^2 R^2).$$

12. A stream of uniform depth and uniform width $2a$ flows slowly through a bridge consisting of two equal arches resting on a rectangular pier of width $2b$, the bridge being so broad that the liquid flows under it with uniform velocity U. Show that after the stream has passed the bridge, the velocity potential of the motion will be

$$(a - b)\, Ux/a + 2Ua/\pi^2 \,.\, \Sigma_1^\infty n^{-2} c^{-n\pi x/a} \sin n\pi b/a \cos n\pi y/a,$$

the axis of x being in the forward direction of the stream, and the origin at the middle point of the pier.

13. Prove that the velocity potential

$$\phi = A\,(\lambda + 2\pi^2 y^2/\lambda) \sin 2\pi\,(vt - x)/\lambda$$

satisfies the equation of continuity in a mass of water, provided the ratio y/λ is so small for all possible values of y that its square may be neglected. Hence prove that if the water in a canal of uniform breadth and uniform depth k, be acted upon in addition to gravity by the horizontal force $Ha^{-1} \sin 2\,(mt - x/a)$ where H and m are small and a is large, the equation of the free surface may be of the form

$$y = k + \frac{Hk}{2\,(gk - m^2 a^2)} \cos 2\,(mt - x/a).$$

14. Prove that in order that indefinite plane waves may be transmitted without alteration with uniform velocity a in a homogeneous fluid medium, the pressure and density must be connected by the equation

$$p - p_0 = a^2 \rho_0^2 (\rho_0^{-1} - \rho^{-1}),$$

where p_0, ρ_0 are the pressure and density in the undisturbed part of the fluid.

15. Two liquids of density ρ, ρ' completely fill a shallow pipe; prove that the velocity of propagation of long waves is

$$U^2 = \frac{g(\rho - \rho')AA'}{b(A'\rho + A\rho')},$$

where A, A' are the areas of the vertical sections of the two liquids when undisturbed, and b is the breadth of the surface of separation.

16. If the upper liquid were moving with mean velocity U', and there is a surface tension T, prove that the wave length is determined by the equation

$$4T\pi^2/\lambda^2 = b(\rho U^2/A + \rho' U'^2/A') - g(\rho - \rho').$$

17. A rectangular pipe whose faces are horizontal and vertical planes, is completely filled with $n+1$ liquids; show that the velocities of propagation of waves of length λ at the surfaces of separation of the strata are given by the equation

$$\begin{vmatrix} A_1 & -B_2 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -B_2 & A_2 & -B_3 & \dots & \dots & \dots & \dots & \dots & \dots \\ & -B_3 & A_3 & -B_4 & \dots & \dots & \dots & \dots & \dots \\ & & -B_4 & A_4 & -B_5 & & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & & & -B_{n-1} & A_{n-1} & -B_n \\ & & & & & & \dots & -B_n & A_n \end{vmatrix} = 0,$$

where

$$A_m = 2\pi v^2/\lambda\,(\rho_{m+1}\coth 2\pi h_{m+1}/\lambda + \rho_m \coth 2\pi h_m/\lambda) - g(\rho_{m+1} - \rho_m)\ B_m$$
$$= 2\pi v^2/\lambda \operatorname{cosech} 2\pi h_m/\lambda$$

and h_m is the equilibrium thickness of the stratum ρ_m.

In particular if $\rho_m = m\sigma$, and $h_m = ma$, then the $2n$ values of v are included in the formula

$$v = \pm \tfrac{1}{2}(ga)^{\frac{1}{2}} \sec \tfrac{1}{2}m\pi/(n+1),$$

where m is supposed to assume the values 1, 2, 3 ..., n, and λ the wave length is supposed very large compared with na.

18. If there be an infinite film in a horizontal plane, separating two heavy liquids of considerable depths, which are flowing in the same directions with velocities V, V' respectively between two

horizontal planes, prove that the velocity of propagation of waves of length λ in the direction of the stream, is given by

$$\sigma (v - V)^2 + \sigma' (v - V')^2 = (\sigma' - \sigma)\, g\lambda/2\pi + 2\pi T/\lambda,$$

where σ, σ' are the densities of the upper and lower liquids respectively, and T is the tension of the film.

19. If the bottom of a horizontal canal of depth h be constrained to execute a simple harmonic motion, such that the vertical displacement at a distance x from a given line across the canal and perpendicular to its length, be given by $k \cos m (x - vt)$, k being small; show that when the motion is steady, the form of the free surface is given by

$$y = h + \frac{kv^2}{v^2 - gh} \cos m (x - vt).$$

20. A mass M of liquid is at rest under the action of its surface tension T. Show that if it be thrown into small vibrations of the type of a zonal harmonic of order n, the time of a small vibration will be

$$\left\{\frac{3\pi M}{n(n-1)(n+2)T}\right\}^{\frac{1}{2}}.$$

21. Prove that upon a shore sloping at an angle $\frac{1}{6}\pi$ below the horizon, a possible state of fluid motion is represented by the velocity potential

$$\phi = A \sin n\pi t \{\epsilon^{-az} \sin ax - \sqrt{3}\epsilon^{-\frac{1}{2}a(z+x\sqrt{3})} \cos \tfrac{1}{2}a (z\sqrt{3} - x)$$
$$+ \epsilon^{\frac{1}{2}a(z-x\sqrt{3})} \sin \tfrac{1}{2}a (z\sqrt{3} + x)\}$$

and that the corresponding current function is

$$\psi = A \sin n\pi t \{\epsilon^{-az} \cos ax - \sqrt{3}\,\epsilon^{-\frac{1}{2}a(z+x\sqrt{3})} \sin \tfrac{1}{2}a (z\sqrt{3} - x)$$
$$- \epsilon^{\frac{1}{2}a(z-x\sqrt{3})} \cos \tfrac{1}{2}a (z\sqrt{3} + x).$$

Prove also that if the motion is small and takes place under the action of gravity,

$$ga = n^2\pi^2.$$

22. A shallow trough is filled with oil and water, the depth of the water being k and its density σ, and that of the oil being h and its density ρ. Prove that the velocity of propagation v of long waves is

$$v^2/g = \tfrac{1}{2}(h + k) + \tfrac{1}{2}\{(h - k)^2 + 4hk\rho/\sigma\}^{\frac{1}{2}}.$$

(Note that there may be slipping between the oil and water.)

23. If water is flowing with velocity proportional to the distance from the bottom, V being the velocity of the stream at its surface, prove that the velocity of propagation U of waves in the direction of the stream is given by

$$(U-V)^2 - V(U-V)\,W^2/gh - W^2 = 0,$$

where W is the velocity of propagation of waves in still water.

24. Suppose that an expanse of liquid is originally still, and plane aerial vibrations of wave length λ and velocity v in air of density ρ', to impinge on the surface at an angle β; prove that when the motion of the system has become periodic, we may represent the displacement of the incident and reflected waves of air, and the displacement of the surface by

$$\text{(i)}\quad a\sin\{m(x\sin\beta + z\cos\beta) - nt - \alpha\},$$

$$\text{(ii)}\quad a\sin\{m(x\sin\beta - z\cos\beta) - nt + \alpha\},$$

$$\text{(iii)}\quad b\cos(mx\sin\beta - nt),$$

respectively, where $m = 2\pi/\lambda$, $n = 2\pi v/\lambda$: prove also that α the change of phase is given by

$$\rho'\cot\alpha = \left(\frac{2\pi T}{\lambda v^2}\sin^2\beta + \frac{g\lambda\rho}{2\pi v^2}\right)\cos\beta - \rho\cot\beta\coth(2\pi h\lambda^{-1}\sin\beta),$$

where T is the surface tension.

25. Prove that with cylindrical coordinates ϖ, θ, z, a possible state of liquid motion inside a right circular cone of vertical angle 2α is given by the velocity potential

$$\phi = Az\varpi^n\cos n\theta\cos 2\pi pt,$$

where $n = \tan^2\alpha$, and that if the axis of the cone be vertical and h be the mean depth of the liquid, the frequency p of such wave motion supposed of small displacement, is given by

$$4\pi^2p^2h = g.$$

26. Two liquids of densities ρ, ρ' each of which half fills a pipe of which the cross section is a square with a vertical diagonal of length $2h$, are slightly disturbed. Neglecting the disturbing effect of the boundary in the neighbourhood of the surface of separation, prove that the velocity of propagation of progressive waves along the pipe is given by the equation

$$U^2 = \frac{g}{2m}\frac{(\rho-\rho')}{(\rho+\rho')}\,(\tanh\ or\ \coth)\,mh.$$

27. A soap bubble of finite thickness in free space with air inside it, is performing small oscillations radially under the action of its surface tension T and the pressure of the contained air. Prove that the length l of the simple equivalent pendulum for vibrations so slow that the contained air may be supposed to obey Boyle's law, is given by the equation

$$3Ma^3b^3g = 8l\pi T(b^2 + ab + a^2)(2b^2 + ab + a^2)(b^2 + ab - a^2),$$

where a and b are the internal and external radii of the shell and M its mass.

28. Prove that in the case of standing waves across a canal of triangular section, whose sides slope at an angle $\frac{1}{6}\pi$ to the horizon, the equation of continuity and the boundary conditions are satisfied by taking

$$\phi = \cos t\,(g/l)^{\frac{1}{2}}\{\sinh m(z-\alpha)\cos mx - \sinh \tfrac{1}{2}m(x\sqrt{3} + z + 2\alpha)$$
$$\times \cos \tfrac{1}{2}m(x - z\sqrt{3}) + \sinh \tfrac{1}{2}m(x\sqrt{3} - z - 2\alpha)\cos \tfrac{1}{2}m(x + z\sqrt{3})\},$$

the axis of x being measured across the canal, and the origin being taken in the line of intersection of the sides.

Prove also that if h be the depth of the canal, ml, α and mh are determined by the equations

$$ml = \tanh m(h-\alpha),\quad 1 - m^2l^2 = ml\sqrt{3}\cot mh\sqrt{3},$$

and one or other of the equations

$$\cosh 3mh = -\cos mh\sqrt{3} + 2\sec mh\sqrt{3},$$

$$3\cosh 3mh = -\cos mh\sqrt{3} - 2\sec mh\sqrt{3}.$$

CHAPTER XVIII.

STABLE AND UNSTABLE MOTION[1].

416. In Chapters XIII. and XIV. we came across several instances in which vortex sheets and motions involving surfaces of discontinuity are unstable; and there is a considerable amount of evidence which supports the conclusion that *when no forces are in action*, all motions involving vortex sheets are unstable. No general proof of this proposition appears as yet to have been given; and it is important to observe that it certainly is not universally true when the liquid is acted upon by any external forces. This may at once be shown by considering the waves propagated at the surface of separation of two liquids, which when undisturbed are moving with velocities V, V'.

Putting k, k' for $m \coth mh$ and $m \coth mh'$, we have shown in § 391 that the velocity of propagation is given by the equation

$$k\rho (V - U)^2 + k'\rho' (V' - U)^2 = g (\rho - \rho').$$

The condition of stability is that the roots of this quadratic in U should be real, and is therefore

$$g (k\rho + k'\rho') (\rho - \rho') - kk'\rho\rho' (V - V')^2 > 0.$$

It therefore follows that if $\rho > \rho'$, that is if the lower liquid is denser than the upper liquid, the motion *may be stable*; but if no forces are in action so that $g = 0$, the motion will be unstable.

[1] This chapter is taken from the following three papers by Lord Rayleigh,

"On the Instability of Jets," *Proc. Lond. Math. Soc.* vol. x.

"On the Stability or Instability of certain Fluid Motions," *Proc. Lond. Math. Soc.*, vols. XI. and XIX.

417. If no forces are in action and both liquids are of unlimited extent so that $h = h' = \infty$, the equation for determining U becomes

$$\rho (V - U)^2 + \rho' (V' - U)^2 = 0 \quad\text{..................(1).}$$

The initial form of the surface of separation is $\eta = a \sin mx$, where m is a real quantity, and its form at any subsequent time is determined by the equation $\eta = a \sin (mx - nt)$.

The roots of (1) are

$$U = \frac{\rho V + \rho' V' \pm \iota \sqrt{\rho\rho'}\,(V - V')}{\rho + \rho'},$$

hence U and therefore n is a complex quantity. Putting

$$U = \alpha + \iota\beta = n/m,$$

and rejecting the imaginary part, the equation of the surface of separation becomes

$$\eta = b \sin m (x - \alpha t) \cosh m\beta t,$$

which indicates that the motion is unstable. The rejected imaginary part shows that if the initial form of this surface was $\eta = b \cos mx$, its equation at any subsequent time would be

$$\eta = b \cos m (x - \alpha t) \cosh m\beta t.$$

There are three cases worthy of notice.

(i) If $\rho = \rho'$, $V = - V'$, so that the densities of the two liquids are equal, and their undisturbed velocities are equal and opposite, $\alpha = 0$, $\beta = V$, whence

$$\eta = b \cosh mVt \sin mx.$$

(ii) Let $\rho = \rho'$, $V' = 0$, then $\alpha = \frac{1}{2}V$, $\beta = \pm \frac{1}{2} V$, and

$$\eta = b \cosh \tfrac{1}{2} mVt \sin m (x - \tfrac{1}{2} Vt),$$

hence the waves travel in the direction of the stream, and with half its velocity.

(iii) Let $\rho = \rho'$, $V = V'$. In this case the roots are equal, but the general solution may be obtained by putting $V' = V(1 + \gamma)$ where γ ultimately vanishes; we thus obtain

$$\eta = b \sin m (x - Vt) \cosh \tfrac{1}{2} V\gamma t - \iota b \cos m (x - Vt) \sinh \tfrac{1}{2} V\gamma t.$$

Putting $\frac{1}{2}\iota b V\gamma = c$, and proceeding to the limit we obtain

$$\eta = b \sin m (x - Vt) - ct \cos m (x - Vt).$$

If $\dot{\eta} = 0$, when $t = 0$, we must have $mbV = -c$, whence

$$\eta = b \sin m(x - Vt) + bmVt \cos m(x - Vt).$$

The peculiarity of this solution is, that previously to displacement there is no real surface of separation at all. Hence if we have a thin surface such as a flag, whose inertia may be neglected, dividing the air, it appears from the last equation that (neglecting changes in the density of the air) the motion of the flag will be unstable, and that it will flap.

418. We shall now investigate the motion of a jet of density ρ and width $2l$, which is flowing with velocity V, and is surrounded by fluid of density ρ' which is at rest.

In solving problems of this class, it is often convenient to employ complex expressions, and in our final results to reject the imaginary parts; we shall therefore suppose that both the surfaces of separation are represented by an equation of the form

$$\eta = a\epsilon^{\iota mx + \iota nt} + l.$$

This is equivalent to supposing that the disturbance is such that the sinuosities of the two surfaces of the jet are parallel.

Let the velocity potential of the jet be

$$\phi = (A \cosh mz + B \sinh mz)\, \epsilon^{\iota mx + \iota nt} + Vx,$$

and that of the surrounding liquid on the upper side be

$$\phi' = C\epsilon^{\iota mx + \iota nt - mz + ml}.$$

The kinematical conditions at the surfaces of separation give

$$A = 0, \quad B = \iota a(n + mV)/m \cosh ml, \quad C = -\iota na/m.$$

The dynamical condition of equality of pressure gives

$$\rho B(n + mV) \sinh ml - \rho' Cn = 0,$$

whence $$\rho(n + mV)^2 \tanh ml + n^2 \rho' = 0.$$

The values of n determined by this equation are always complex unless ρ' is zero. When $\rho = \rho'$,

$$n = \frac{-mV \tanh ml \pm \iota mV (\tanh ml)^{\frac{1}{2}}}{1 + \tanh ml}.$$

When ml is small, we have approximately

$$\eta = a\epsilon^{\pm (ml)^{\frac{1}{2}} Vmt} \cos m(Vmlt - x).$$

419. The motion of a straight cylindrical jet, whose cross section is a circle, and which is surrounded by liquid which is at rest, has also been investigated by Lord Rayleigh, and the results are similar to those already obtained for a two-dimensional jet. If z be measured along the axis of the jet, the displacement of any point on its surface can be shown to be

$$\varpi = a\epsilon^{\pm\mu Vt} \cos m\,(Vt - z),$$

where $$\mu^2 = m^2a^2 \{\log 8/ma + \pi^{-\frac{1}{2}}\,\Gamma'(\tfrac{1}{2})\}.$$

420. It is a matter of observation, that when a jet of water issues continuously from a small orifice, the continuity of the liquid ceases at a certain distance from the orifice, and the jet becomes disintegrated into drops. The preceding investigations partially explain this phenomenon, since the jet is necessarily surrounded by air, and we have shown that the motion in such a case must be unstable. It must however be admitted that the results obtained are only rough approximations, since we have supposed (i) that the air by which the jet is surrounded is incompressible and at rest, (ii) that the liquid of which the jet is composed is free from viscosity, (iii) we have neglected the existence of capillarity at its surface. When we consider the motion of a viscous liquid, it will be shown that a surface of discontinuity, if it ever could be formed, would instantly disappear, and that molecular rotation would be propagated on either side of the surface according to the law of propagation of heat. Hence our results are necessarily imperfect. We shall return to this point hereafter; and shall now proceed to investigate the effect of surface tension on a cylindrical jet moving in vacuo.

421. Taking the axis of z along the axis of the cylinder, let us suppose that the surface of the jet at time t is

$$r = a + \alpha \cos \kappa z,$$

where α is a small function of the time, and $\kappa = 2\pi/\lambda$.

Let σ be the area of the surface of the jet included between unit of length; then

$$\sigma = 2\pi\lambda^{-1} \int_0^\lambda (a + \alpha \cos 2\pi z/\lambda + \tfrac{1}{2}a\alpha^2\kappa^2 \sin^2 2\pi z/\lambda)\, dz$$

$$= \pi a\,(2 + \tfrac{1}{2}\kappa^2\alpha^2) \quad\text{...(2)}$$

approximately. In this expression a is not absolutely constant;

its value is determined from the fact that the volume V included between unit length is constant, whence

$$V = \pi a^2 + \tfrac{1}{2}\pi \alpha^2 \dots\dots\dots\dots\dots\dots\dots\dots (3).$$

Now (2) may be written

$$\sigma = 2\pi a + \tfrac{1}{2}\pi \alpha^2/a + \tfrac{1}{2}\pi \alpha^2 (\kappa^2 a^2 - 1)/a.$$

Substituting the value of a from (3) in the first two terms we obtain

$$\sigma = 2(\pi V)^{\frac{1}{2}} + \tfrac{1}{2}\pi \alpha^2 (\kappa^2 a^2 - 1)/a.$$

If σ_0 be the value of σ for the undisturbed motion, we have

$$\sigma - \sigma_0 = \tfrac{1}{2}\pi \alpha^2 (\kappa^2 a^2 - 1)/a \dots\dots\dots\dots\dots\dots (4).$$

If T_1 denote the surface tension, the potential energy per unit of length from the position of equilibrium is

$$V = -\tfrac{1}{2}\pi T_1 \alpha^2 (1 - \kappa^2 a^2)/a \dots\dots\dots\dots\dots\dots (5).$$

Since the motion is symmetrical with respect to the axis of z, Laplace's equation is

$$\frac{d^2\phi}{dz^2} + \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = 0,$$

and since ϕ must vary as $\cos \kappa z$, the proper solution is

$$\phi = A I_0 (\kappa r) \cos \kappa z.$$

The coefficient A is determined from the fact that the normal velocity at the surface of the jet is equal to $\dot{\alpha} \cos \kappa z$, whence

$$A \kappa I'_0 (\kappa a) = \dot{\alpha},$$

and therefore

$$\phi = \frac{\dot{\alpha} I_0 (\kappa r)}{\kappa I'_0 (\kappa a)} \cos \kappa z.$$

Taking the density of the liquid to be unity, the kinetic energy per unit of length is

$$T = (2\lambda)^{-1} \int_0^\lambda 2\pi a \phi (d\phi/dr)_a \, dz$$

$$= \tfrac{1}{2}\pi a^2 \dot{\alpha}^2 \frac{I_0 (\kappa a)}{\kappa a I'_0 (\kappa a)},$$

whence by (5) the equation of motion is

$$\frac{a^3 \dot{\alpha}^2 I_0 (\kappa a)}{\kappa a I'_0 (\kappa a)} - T_1 \alpha^2 (1 - \kappa^2 a^2) = \text{const.}$$

Differentiating with respect to t, and then putting $\alpha = A\epsilon^{qt}$, we obtain

$$q^2 = \frac{T_1 (1 - \kappa^2 a^2)\, \kappa a I'_0 (\kappa a)}{a^3 I_0 (\kappa a)}.$$

If $\kappa a > 1$, q is imaginary, and the motion is stable; hence from (4) it follows that if the surface is greater after displacement than before, the motion is stable; but if otherwise the motion is unstable. Writing $\kappa a = x$, the instability will be greatest when λ has such a value that q is a maximum.

Since

$$I_0 (x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} + \frac{x^2}{2^2 . 4^2 . 6^2} +$$

the value of $q^2 a^3/T_1$ will be found to be

$$\tfrac{1}{2} x^2 (1 - x^2) \left\{1 - \frac{x^2}{2^3} + \frac{x^4}{2^4 . 3} - \frac{11 x^6}{2^{10} . 3} + \frac{19 x^8}{2^{11} . 3 . 5} + \ldots\right\},$$

or

$$\tfrac{1}{2} \left\{x^2 - \tfrac{9}{8} x^4 + \frac{7}{2^4 . 3} x^6 - \frac{25}{2^{10}} x^8 + \frac{91}{2^{11} . 3 . 5} x^{10} + \ldots\right\}.$$

Differentiating we obtain

$$1 - \tfrac{9}{4} x^2 + \frac{7}{2^4} x^4 - \frac{100}{2^{10}} x^6 + \frac{91}{2^{11} . 3} x^8 + \ldots = 0.$$

If all the terms but the first three be neglected, the quadratic gives $x^2 = {\cdot}4914$; and if this value be substituted in the next two terms, the equation becomes

$${\cdot}98928 - \tfrac{9}{4} x^2 + \tfrac{7}{16} x^4 = 0,$$

whence

$$x^2 = {\cdot}4858.$$

The corresponding value of λ is given by

$$\lambda = 4{\cdot}508 \times 2a,$$

which gives the ratio of the wave length to the diameter, for the kind of disturbance which leads most rapidly to the disintegration of the cylindrical mass. The corresponding number obtained by Plateau from some experiments by Savart is 4·38, but as his estimate involves a knowledge of the coefficient of contraction of a jet escaping through a small hole in a thin plate, it is probably liable to a greater error than its deviation from 4·51.

Further information on the subject of jets in connection with hydraulic machinery, will be found in Prof. W. C. Unwin's article on *Hydraulics*, in the *Encyclopaedia Britannica*.

Stability of Steady Motion between Two Parallel Planes[1].

422. Let the liquid be bounded by two parallel planes, and be moving with velocity U parallel to those planes; also let the axis of x lie in one of the planes, and be parallel to the direction of U. If the motion is steady, U must be a function of y alone, and the vorticity $\zeta = -\frac{1}{2}dU/dy$.

Let a disturbance of any kind be communicated to the liquid, subject only to the condition that the resulting motion is in two dimensions; and let $U+u$, v, $\zeta+\zeta'$ be the component velocities and vorticity during the disturbed motion. Then

$$\frac{d\zeta'}{dt} + U\frac{d\zeta'}{dx} + v\frac{d\zeta}{dy} = 0,$$

$$\frac{du}{dx} + \frac{dv}{dy} = 0, \qquad 2\zeta' = \frac{dv}{dx} - \frac{du}{dy}.$$

If we assume that x and t enter into u, v, ζ' in the form of the factor $\exp(\iota nt + \iota kx)$, the preceding equations may be written

$$\iota(n+kU)\zeta' = \tfrac{1}{2}v d^2U/dy^2, \quad \iota\kappa u + dv/dy = 0, \quad 2\zeta' = \iota kv - du/dy.$$

Eliminating u and ζ' we obtain,

$$\left(\frac{n}{k} + U\right)\left(\frac{d^2v}{dy^2} - k^2v\right) = \frac{d^2U}{dy^2}v \quad\text{................(6).}$$

423. We must now determine the boundary conditions.

At the surfaces of the bounding planes we must have $v = 0$. It may also happen that the vorticity in steady motion suddenly changes as we cross some plane, and we must therefore find the conditions to be satisfied at the surface of separation. Denoting by Δ the difference between the values of the quantities on either side of this surface, the kinematical condition is

$$\Delta v = 0 \quad\text{..................................(7).}$$

The dynamical condition which is the analytical expression for the fact that there must be no discontinuity of pressure, may be obtained by integrating (6) across the surface; we thus obtain

$$\left(\frac{n}{k} + U\right)\Delta\frac{dv}{dy} - v\Delta\frac{dU}{dy} = 0 \quad\text{................ (8).}$$

[1] Lord Rayleigh, *Proc. Lond. Math. Soc.* vols. XI. and XIX.

424. We shall now apply these equations to determine the conditions of stability of a mass of liquid bounded by the planes $y=0$, $y=a+b+c$, and which consists of three layers of thickness a, b, c, the vorticity being constant but different throughout each layer.

Let $U=0$ along Ox, and let U_1, U_2 be the values of U at the planes $y=a$, $y=a+b$. Since ζ is constant, $d^2U/dy^2=0$, hence if $n/k+U$ is not zero, (6) becomes

$$\frac{d^2v}{dy^2}-k^2v=0,$$

the solution of which is

$$v=A\cosh ky+B\sinh ky.$$

Since $v=0$ when $y=0$, we must have at the first layer

$$v=v_1=\sinh ky \quad\text{.........................(9)},$$

in the second

$$v=v_2=v_1+M_1\sinh k(y-a) \quad\text{..............(10)},$$

and in the third

$$v=v_3=v_2+M_2\sinh k(y-a-b) \quad\text{...........(11)}.$$

The condition that $v=0$ when $y=a+b+c$, gives

$$M_2\sinh kc+M_1\sinh k(b+c)+\sinh k(a+b+c)=0 \quad\text{...(12)}.$$

If we denote the values of $\Delta dU/dy$ at the two surfaces by Δ_1 and Δ_2 respectively, the condition (8) gives

$$(n+kU_1)M_1-\Delta_1\sinh ka=0 \quad\text{..............(13)},$$

$$(n+kU_2)M_2-\Delta_2\{M_1\sinh kb+\sinh k(a+b)\}=0 \quad\text{...(14)}.$$

Eliminating M_1, M_2 between (12), (13) and (14), we shall find that n satisfies the quadratic

$$An^2+Bn+C=0 \quad\text{.....................(15)},$$

where

$$\left.\begin{aligned}
A&=\sinh k(a+b+c)\\
B&=k(U_1+U_2)\sinh k(a+b+c)+\Delta_1\sinh ka\sinh k(b+c)\\
&\qquad+\Delta_2\sinh kc\sinh k(a+b)\\
C&=k^2U_1U_2\sinh k(a+b+c)+kU_1\Delta_2\sinh kc\sinh k(a+b)\\
&\quad+kU_2\Delta_1\sinh ka\sinh k(b+c)+\Delta_1\Delta_2\sinh ka\sinh kb\sinh kc
\end{aligned}\right\}\text{...(16)}.$$

The condition that the roots of (15) should be real, is that $B^2 - 4AC$ should be positive. Now,

$$B^2 - 4AC = \{k(U_1 - U_2)\sinh k(a+b+c) + \Delta_1 \sinh ka \sinh k(b+c) \\ - \Delta_2 \sinh kc \sinh k(a+b)\}^2 + 4\Delta_1\Delta_2 \sinh^2 ka \sinh^2 kc \ldots\ldots(17).$$

If therefore Δ_1, Δ_2 have the same sign, so that the curve expressing U as a function of y is of one curvature throughout, the roots are real and therefore the disturbed motion is stable.

425. Let us now suppose that the breadths of the layers a and c are equal, and that their vorticities are equal and opposite, and that the layer b is without vorticity; also let V be the velocity of the middle layer. If we suppose the velocity of the liquid to be zero at the walls, which we may do without loss of generality, we shall have

$$U_1 = U_2 = V, \quad \Delta_1 = \Delta_2 = - V/a,$$

whence $$B^2 - 4AC = 4\Delta^2 \sinh^4 ka,$$

indicating stability. Also

$$n + kV = \frac{V\{\sinh ka \sinh k(a+b) \pm \sinh^2 ka\}}{a \sinh k(2a+b)},$$

which determines the relation between n and k.

426. In the next place let us suppose that the velocities are equal and opposite on either side of the middle layer; then the velocities in steady motion, in the first, second and third layers will be respectively

$$v_1 = A(y-a) + V, \quad v_2 = V(1 - 2y/b + 2a/b), \quad v_3 = E(y-a-b) - V,$$

also if the velocities at the bounding planes are equal and opposite we must have $A = E$. We thus obtain

$$U_1 = - U_2 = V, \quad \Delta_1 = - \Delta_2 = \mu V,$$

where $\mu = -A/V - 2b^{-1}$. From (16) it follows that $B = 0$, and (15) may be written

$$\frac{n^2}{k^2 V^2} = \frac{\{\mu \sinh x \sinh y + k \sinh(x+y)\}^2 - k^2 \sinh^2 x}{k^2 \sinh y \sinh(2x+y)},$$

where $x = ka$, $y = kb$.

From this expression it is easily seen that n^2 is positive if μ is positive; but if μ is negative the motion will be unstable unless

the numerator of the above fraction is positive. Writing $-\nu$ for μ, this requires that

$$\{k(\coth x + \coth \tfrac{1}{2}y) - \nu\}\{k(\coth x + \tanh \tfrac{1}{2}y) - \nu\} > 0 \ldots (18).$$

If we suppose that k is very small this becomes

$$(a^{-1} + 2b^{-1} - \nu)(a^{-1} - \nu) > 0.$$

Hence if $\nu < a^{-1}$ the motion is stable, but if $a^{-1} + 2b^{-1} > \nu > a^{-1}$, the motion is unstable.

When $\nu = a^{-1} + 2b^{-1}$ the motion is on the border line between stability and instability, but it is really unstable; for proceeding to a second approximation the first factor of (18) becomes

$$a^{-1} + 2b^{-1} - \tfrac{1}{3}k^2 a - \tfrac{1}{6}k^2 b - \nu,$$

which shows that the motion is unstable. Now if U be the velocity of the liquid in contact with the plane $y = 0$, $U = V - Aa$, whence

$$\mu = -(V - U)/Va - 2b^{-1}.$$

Hence the final condition of complete stability is that

$$2Ub > Va.$$

Steady Motion between Two Concentric Cylinders[1].

427. We shall now prove that if liquid is in motion between two rigid concentric circular cylinders, the steady motion is stable provided the vorticity either continually increases or continually decreases in passing outward from the axis.

In steady motion let V be the velocity and ω the vorticity, then V and ω are functions of r alone. Let u, $V + v$ be the velocities along and perpendicular to the radius vector during the disturbed motion, $\omega + \zeta$ the vorticity.

These quantities satisfy the following equations:

$$2\omega = \frac{dV}{dr} + \frac{V}{r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

$$\frac{d\zeta}{dt} + u\frac{d\omega}{dr} + \frac{V}{r}\frac{d\zeta}{d\theta} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (20),$$

$$\frac{d(ru)}{dr} + \frac{dv}{d\theta} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (21),$$

$$\frac{dv}{dr} + \frac{v}{r} + \frac{1}{r}\frac{du}{d\theta} = 2\zeta \ldots\ldots\ldots\ldots\ldots\ldots (22).$$

[1] Lord Rayleigh, *Proc. Lond. Math. Soc.* vol. XI.; see also Sir W. Thomson, On Maximum and Minimum Energy in Vortex Motion, *Phil. Mag.* June 1887.

Equation (19) is the equation connecting the velocity and vorticity in steady motion, and (20), (21) determine the changes in the velocity and vorticity due to the disturbance.

Let us assume that u, v, ζ are each of the form $F(r)\exp(\iota k\theta + \iota nt)$ where k is a real quantity but n may be complex. We have to determine the conditions that n may be real. From (20) and (22) we obtain

$$\tfrac{1}{2}\iota\left(n + \frac{kV}{r}\right)\left(\frac{dv}{dr} + \frac{v}{r} + \frac{\iota\kappa u}{r}\right) + u\frac{d\omega}{dr} = 0,$$

and from (21)

$$\frac{d(ru)}{dr} + \iota kv = 0.$$

Putting $ru = p$ and eliminating v, we obtain

$$\left(\frac{n}{k} + \frac{V}{r}\right)\left(\frac{d^2p}{dr^2} + \frac{1}{r}\frac{dp}{dr} - \frac{k^2p}{r^2}\right) = \frac{2p}{r}\frac{d\omega}{dr}.$$

Let $n/k = e + \iota f$, $p = \alpha + \iota\beta$, where e, f, α, β are real, then

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{k^2}{r^2}\right)(\alpha + \iota\beta) = 2\frac{d\omega}{dr}\frac{(\alpha + \iota\beta)}{(e + \iota f)r + V}.$$

Whence equating the real and imaginary parts we obtain

$$\frac{d^2\alpha}{dr^2} + \frac{1}{r}\frac{d\alpha}{dr} + \frac{k^2\alpha}{r^2} = \frac{2}{D^2}\{\alpha(V + re) + \beta f\}\frac{d\omega}{dr},$$

$$\frac{d^2\beta}{dr^2} + \frac{1}{r}\frac{d\beta}{dr} + \frac{k^2\beta}{r^2} = \frac{2}{D^2}\{\beta(V + re) - \alpha f\}\frac{d\omega}{dr},$$

where $D^2 = (V + re)^2 + r^2f^2$. Multiplying the first equation by $r\beta$ and the second by $r\alpha$ and subtracting we obtain

$$r\left(\beta\frac{d^2\alpha}{dr^2} - \alpha\frac{d^2\beta}{dr^2}\right) + \beta\frac{d\alpha}{dr} - \alpha\frac{d\beta}{dr} = \frac{2fr}{D^2}(\alpha^2 + \beta^2)\frac{d\omega}{dr}.$$

Let a and b be the radii of the cylindrical boundaries, then since α and β are each zero at the boundaries, we obtain on integrating between the limits a and b,

$$2f\int_b^a \frac{r}{D^2}(\alpha^2 + \beta^2)\frac{d\omega}{dr}r = 0.$$

If $d\omega/dr$ does not change sign between the limits, every element of this integral has the same sign, and therefore the integral cannot vanish unless $f = 0$; when this is the case n is real and therefore the steady motion is stable.

CHAPTER XIX.

THE THEORY OF THE TIDES.

428. THE phenomenon of the tides is produced, as is well known, by the disturbing attractions of the sun and moon upon the ocean. This appears to have been first recognised by Kepler, but the subject was not investigated mathematically until the year 1687, when Newton[1] applied the law of gravitation to the explanation of the tides. He supposed that the ocean covers the whole earth, and that it assumes at each instant a figure of equilibrium under the combined attractions of the earth, sun and moon. In 1738 Daniel Bernoulli[2] extended and improved Newton's theory, and the theory of the former is usually known as the *Equilibrium Theory*. This theory although it serves to explain many of the principal features of the tides, cannot be considered satisfactory; for the problem is essentially a dynamical one, and consists in finding the forced oscillations of an ocean which is disturbed by the attractions of the sun and moon. The solution of the dynamical problem was first effected by the genius of Laplace[3], upon the assumptions that the ocean covers the whole earth, and that its depth is equal to $l(1 - q\cos^2\theta)$, where θ is the co-latitude, and l and q are constants. The original investigation of Laplace is however unnecessarily complicated by the use of spherical harmonic analysis; it was subsequently presented in a simpler form by Airy[4], but the investigation of the latter contains a criticism on Laplace's method of dealing with a certain continued fraction, which occurs

[1] *Principia*, Book I. Prop. 66, Cor. 19; Book III. Props. 26 and 27.

[2] *Acad. des Sciences*, Paris, 1738.

[3] *Mécanique Céleste*, Book IV.

[4] "Tides and Waves," *Encyc. Met.* Sec. III.

in the evaluation of the semi-diurnal tide in an ocean of uniform depth, which is now generally considered to be erroneous. Laplace's procedure was justified by Sir W. Thomson[1], and the controverted point has been fully worked out and explained by Prof. G. H. Darwin[2], and it is from the papers of the latter that the following investigation of Laplace's theory is taken. A third theory, known as the *Canal Theory*, which is due to Airy[3], consists in investigating the tides in a canal coinciding with a small circle upon the earth, which are produced by a disturbing body revolving about the earth in an orbit, whose projection upon the earth's surface is a different small circle.

In the present chapter we shall discuss these three theories[4].

The Equilibrium Theory.

429. In the equilibrium theory, the earth is supposed to consist of a solid spherical nucleus, whose density is either uniform or which is composed of spherical strata of uniform density. The solid nucleus is supposed to be covered with water, which is disturbed by the attractions of the sun and moon; and it is required to find the form of the free surface of the water, on the supposition that at every instant it assumes the form of a surface of equilibrium under the combined attractions of the earth, sun and moon.

Since the disturbing attractions of the sun and moon are both small in comparison with that of the earth, we may consider the effects of each luminary separately, and the combined effect of both will be obtained by adding the effects due to each.

Whether the disturbing body is the sun or moon, the earth may be supposed to be reduced to rest by including amongst the forces which act upon the ocean, the reversed acceleration of the

[1] *Phil. Mag.* 1875.

[2] "Tides," *Encyc. Brit.; Proc. Roy. Soc.* 1886.

[3] "Tides and Waves," *Encyc. Met.* Sec. VI.

[4] For further information, see

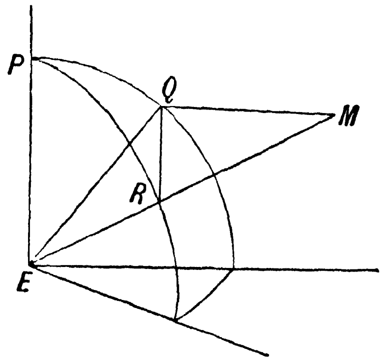
Bibliographie de l'Astronomie, by Houzeau and Lancaster, Brussels, 1882 which contains a complete list of works upon the subject down to 1881;

Thomson and Tait, vol. I. part II.;

Reports on Tides to the British Association;

Thomson, *Phil. Mag.* 1880.

centre of the earth towards the disturbing body. This reversed acceleration is equal to the mass of the disturbing body divided by the square of its distance from the centre of the earth. We shall also suppose that the rotation of the earth is annulled, and that the disturbing body revolves round the earth.



Let E be the centre of the earth, P its pole, Q any point of the ocean. Let M be the moon, $EM = D$, $ER = r$, $MEQ = \epsilon$; also let V be the attraction potential of all the forces which act on the ocean, and let v be the potential of the earth and the ocean.

Resolving the force upon an element of liquid at Q along EQ, we obtain

$$\frac{dV}{dr} = \frac{M}{MQ^2}\cos(\pi - EQM) + \frac{dv}{dr} - \frac{M}{D^2}\cos\epsilon,$$

$$= \frac{M(D\cos\epsilon - r)}{(r^2 + D^2 - 2Dr\cos\epsilon)^{\frac{3}{2}}} + \frac{dv}{dr} - \frac{M}{D^2}\cos\epsilon,$$

whence $$V = \frac{M}{(r^2 + D^2 - 2Dr\cos\epsilon)^{\frac{1}{2}}} + v - \frac{Mr}{D^2}\cos\epsilon + A \ldots (1).$$

Since the right-hand side of (1) is a potential function, it is unnecessary to add a function of ϵ, and therefore A is an absolute constant. Expanding and neglecting spherical harmonics of a higher degree than the second, we obtain

$$V = \frac{M}{D} + \frac{Mr^2}{D^3} P_2(\cos\epsilon) + v + A.$$

Let a be the radius of the free surface of the ocean when undisturbed, $a + \mathfrak{e}$ its value when disturbed, so that $\mathfrak{e}$ is the height of the tide. Then $\mathfrak{e}$ may be expanded in a series of spherical harmonics whose axis is EM; and since the value of V must be constant at the surface of the ocean, it follows that $\mathfrak{e}$ cannot contain any harmonics of the first degree. Also since the depth of the

ocean is small in comparison with the radius of the earth, it follows from (7) of § 371 that if we neglect harmonics of a higher degree than the second,

$$v = E/r + \tfrac{4}{5}\pi a\sigma\,(a/r)^2\,\mathfrak{r},$$

where E is the mass of the earth, and σ is the density of the ocean. Hence if ρ be the density of the earth, the condition that V should be constant at the surface of the ocean is that

$$\frac{E}{a^2}\left(1 - \frac{3\sigma}{5\rho}\right)\mathfrak{r} = \frac{Ma^2}{D^3}P_2,$$

and since $E/a^2 = g$, we obtain

$$\mathfrak{r} = \frac{Ma^2P_2}{D^3\,(1 - 3\sigma/5\rho)\,g}.$$

This equation determines the height of the tide upon the equilibrium theory, and shows that the form of the free surface at any period is a prolate spheroid, whose longest axis coincides with the line joining the centre of the earth with the disturbing body.

Since the density of the ocean is small in comparison with that of the earth, the quantity σ/ρ is usually neglected, in which case we obtain

$$g\mathfrak{r} = Ma^2P_2/D^3 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

430. Before proceeding to discuss this equation, it should be noticed that owing to the fact that the earth is not entirely covered with water, the value of $\mathfrak{r}$ requires correction. A description of the necessary corrections, together with tables containing the results of observations at various ports, will be found in Thomson and Tait's *Natural Philosophy*, Vol. I., Part II., §§ 808—810 and § 848.

431. Let λ be the latitude and l the west longitude of Q; also let h be the westward hour angle of the moon from Greenwich, δ the moon's declination. Then since the angle $QPR = h - l$, we obtain from the spherical triangle QPR,

$$\cos\epsilon = \sin\lambda\sin\delta + \cos\lambda\cos\delta\cos(h - l),$$

therefore

$$\begin{aligned} P_2(\cos\epsilon) &= \tfrac{1}{2}(3\cos^2\epsilon - 1) \\ &= \tfrac{1}{4}(3\sin^2\delta - 1)(3\sin^2\lambda - 1) + 3\sin\lambda\cos\lambda\sin\delta\cos\delta\cos(h - l) \\ &\quad + \tfrac{3}{4}\cos^2\lambda\cos^2\delta\cos 2(h - l)\ldots\ldots\ldots\ldots\ldots\ldots(3). \end{aligned}$$

A similar equation holds good when the sun is the disturbing body; whence writing S, D', δ', h' for the mass, distance, declination and hour angle of the sun, the height of the tide due to the combined action of the sun and moon is

$$\left.\begin{aligned}\mathfrak{r} = &\frac{a^2}{2g}(1-3\sin^2\lambda)\left\{\frac{M}{D^3}\left(\tfrac{3}{2}\cos^2\delta-1\right)+\frac{S}{D'^3}\left(\tfrac{3}{2}\cos^2\delta'-1\right)\right\}\\ &+\frac{3a^2}{4g}\sin 2\lambda\left\{\frac{M}{D^3}\sin 2\delta\cos(h-l)+\frac{S}{D'^3}\sin 2\delta'\cos(h'-l)\right\}\\ &+\frac{3a^2}{4g}\cos^2\lambda\left\{\frac{M}{D^3}\cos^2\delta\cos 2(h-l)+\frac{S}{D'^3}\cos^2\delta'\cos 2(h'-l)\right\}\end{aligned}\right\}\ldots(4).$$

We shall now proceed to discuss this equation[1].

432. *Tides of Long Period.* The first line of this expression does not depend upon $h-l$ or $h'-l$, and is therefore independent of the hour of the day. It depends solely upon the latitude of the place of observation, and upon the quantities D, D', δ, δ'. The quantities D, δ depend upon the elements of the moon's orbit round the earth, and it will be observed that the value of the first term due to the moon's action does not depend upon the sign of δ, and therefore has the same value whether the moon's declination is north or south. Also since the moon approximately takes fourteen days to describe a semi-circle, the effect of the first term is to produce a *fortnightly tide.*

The second term of the first line is due to the action of the sun; it depends upon the elements of the earth's orbit round the sun, and produces a *semi-annual tide.* Both these tides are known as *tides of long period,* and are called by Laplace, "*Les oscillations de la première espèce.*" They vanish in latitude $\pm$ cosec $\sqrt{3}$.

433. *The Diurnal Tides.* The second line of (4) consists of two terms each of which depends upon the hour angle of the disturbing body. The first term goes through all its changes each time the moon's hour angle increases by 360°; and the second term goes through its period when the sun's hour angle increases by the same amount. These terms constitute the *diurnal tides,* and are called by Laplace, "*Les oscillations de la seconde espèce.*"

[1] Airy, *Tides and Waves.*

Since $h'-l=h-l+h'-h$, the second line of (4) may be written

$$\frac{3a^2}{4g}\sin 2\lambda\left[\left\{\frac{M}{D^3}\sin 2\delta+\frac{S}{D'^3}\sin 2\delta'\cos(h'-h)\right\}\cos(h-l)\right.$$
$$\left.-\frac{S}{D'^3}\sin 2\delta'\sin(h'-h)\sin(h-l)\right]$$
$$=\frac{3a^2}{4g}\sin 2\lambda\left[\frac{M^2}{D^6}\sin^2 2\delta+\frac{S^2}{D'^6}\sin^2 2\delta'\right.$$
$$\left.+\frac{2SM}{D^2D'^3}\sin 2\delta\sin 2\delta'\cos(h'-h)\right]\cos(h-l+E),$$

where $$\tan E=\frac{SD'^{-3}\sin 2\delta'\sin(h'-h)}{MD^{-3}\sin 2\delta+SD'^{-3}\sin 2\delta'\cos(h'-h)}.$$

This tide always vanishes at the equator where $\lambda=0$, and also at the poles where $\lambda=90^\circ$, and its greatest value is in latitude 45°.

For any particular point on a meridian, the tide will be highest when $h-l+E=0$, or $\epsilon+E=0$. Hence if ϵ is positive, that is if the moon is west of the place of observation, E is negative, and therefore if δ' is positive $h'-h$ is negative. Also $\tan E$ and therefore E is always a small quantity, hence just before new moon and full moon high tide occurs shortly after the moon has passed the meridian. But if E is positive ϵ is negative, and therefore just after new moon and full moon high tide occurs before the moon passes the place of observation.

Let us now examine how this tide depends on δ. Since S/D'^3 is small in comparison with M/D^3, it follows that $\tan E$ is positive if δ, δ', $h'-h$ are positive. Now suppose that the moon crosses the equator, then δ will change sign, and E will change rapidly from a small angle through $\frac{1}{2}\pi$ and then to an angle not much less than π. Hence ϵ will rapidly change to $\pi-\epsilon$, and high water instead of occurring when the moon is near the meridian of the place of observation, will take place when the moon is near the meridian passing through the antipodes.

434. *The Semi-diurnal Tides.* The third line of (4) consists of two terms each of which depends upon twice the hour angle of the disturbing body, and goes through all its changes every time the hour angle increases by 180°. These terms constitute the *semi-diurnal tides*, and are called by Laplace "*Les oscillations de la troisième espèce.*"

This term may be expressed in the form

$$\frac{3a^2}{4g}\cos^2\lambda\left\{\frac{M^2}{D^6}\cos^4\delta+\frac{S^2}{D'^6}\cos^4\delta'\right.$$
$$\left.+\frac{2MS}{D^3D'^3}\cos^2\delta\cos^2\delta'\cos 2(h'-h)\right\}\cos\{2(h'-h)+F\},$$

where $$\tan F=\frac{SD'^{-3}\cos^2\delta'\sin 2(h'-h)}{MD^{-3}\cos^2\delta+SD'^{-3}\cos^2\delta'\cos 2(h'-h)}.$$

For different latitudes this tide has its greatest value at the equator where $\lambda=0$, and vanishes at the poles where $\lambda=90^\circ$. For different positions of the moon it is a maximum when $h'-h=0$ or π, that is at full moon and new moon; and it is a minimum when $h'-h=\frac{1}{2}\pi$ or $\frac{3}{2}\pi$, that is when the moon is in quadratures.

The time of high water at any place is found by putting $h'-h=-\frac{1}{2}F$ or $\epsilon=-\frac{1}{2}F$; hence when $\frac{1}{2}\pi>h'-h>0$ and when $\frac{3}{2}\pi>h'-h>\pi$, ϵ is negative, and therefore between new moon and quarter moon, and between full moon and three-quarter moon high tide occurs before the moon has passed the place of observation; and the contrary is the case between quarter moon and full moon and between three-quarter moon and new moon.

Laplace's Theory[1].

435. The celebrated theory of the tides which is due to Laplace, deals with the problem by means of dynamical principles. The problem to be solved, consists in finding a solution of the general equations of motion applicable to the case of the forced oscillations of a frictionless liquid, which completely covers a solid spherical nucleus, and whose equilibrium is disturbed by the attraction of a distant body. The depth of the ocean is supposed to be very small in comparison with the radius of the earth, and the height of the waves small in comparison with their lengths, in other words the waves are assumed to be *long waves* in shallow water, and the equations determining the oscillations of the ocean are obtained; but the mathematical difficulties of integrating them are so great, that Laplace was compelled to assume that the depth of the ocean is proportional to $1-q\cos^2\theta$, where θ is the co-latitude

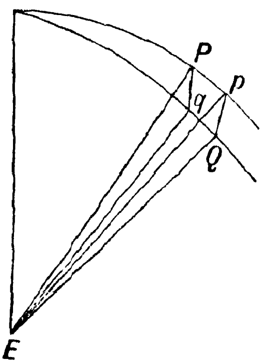
[1] *Mécanique Céleste*, Livres I. and IV.

and q is a constant; and in the case of the semi-diurnal tides it is necessary to suppose that q is zero or unity.

The method of procedure adopted in the present section is somewhat different from Laplace's, and is taken from Prof. G. H. Darwin's[1] investigations, which were suggested by Sir W. Thomson's papers in the *Philosophical Magazine*[2].

436. We must first find a suitable form of the equation of continuity.

Let a be the radius of the earth, γ the depth of the ocean, θ the co-latitude, and ϕ the longitude of any point P on its surface when undisturbed. Let $\theta + u$, $\phi + v$, $a + \gamma + \mathfrak{h}$ be the co-ordinates of the same point during the disturbed state. Let Q be a point near P, whose undisturbed co-ordinates are $a + \gamma$, $\theta + \delta\theta$, $\phi + \delta\phi$.



Let Pp, Qq be the meridians, and Pq, Qp the parallels of latitude passing through P and Q; and consider a small column of liquid which is bounded by the bottom of the ocean, the free surface, and the four planes drawn through E the centre of the earth and the arcs Pp, pQ, Qq, qP respectively.

The volume of this column of liquid when the ocean is undisturbed is

$$a^2\gamma \sin\theta\,\delta\theta\,\delta\phi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The volume of this element when the ocean is disturbed, will be found by changing θ and ϕ into $\theta + u$, $\phi + v$, and hence γ must be changed into $\gamma + \mathfrak{h} + u d\gamma/d\theta + v d\gamma/d\phi$; whence the volume is equal to

$$a^2\left(\gamma + \mathfrak{h} + u\frac{d\gamma}{d\theta} + v\frac{d\gamma}{d\phi}\right)\sin(\theta + u)\left(1 + \frac{du}{d\theta}\right)\left(1 + \frac{dv}{d\phi}\right)\delta\theta\delta\phi \ldots(6),$$

[1] *Proc. Roy. Soc.* 1886; and *Encyc. Brit.*, Art. "Tides."
[2] *Phil. Mag.* 1875.

equating (5) and (6) and neglecting squares and products of small quantities we obtain,

$$\mathfrak{h} + \frac{d(\gamma u)}{d\theta} + \frac{d(\gamma v)}{d\phi} + \gamma u \cot\theta = 0 \;\ldots\ldots\ldots\ldots\; (7).$$

437. In order to obtain the equations of motion of the ocean, we shall reduce the centre of the earth to rest, and we must therefore include amongst the impressed forces which act upon individual particles of liquid, the reversed acceleration of the centre of the earth. This reversed acceleration is equal to the attraction of the sun and moon upon a unit particle situated at the centre of the earth. We must also suppose the sun to be revolving round the earth.

If therefore r, θ', ϕ' are the co-ordinates of an element of the ocean in its disturbed position; w', u', v' the component velocities of this element relative to the centre of the earth in the directions in which r, θ', ϕ' increase, the relative accelerations are

$$\begin{array}{llll} \dot{u}' - v'\theta_3 + w'\theta_2 & \text{in the direction of } \theta', \\ \dot{v}' - w'\theta_1 + u'\theta_3 & \quad ,, \qquad ,, \qquad ,, \;\phi', \\ \dot{w}' - u'\theta_2 + v'\theta_1 & \quad ,, \qquad ,, \qquad ,, \;r. \end{array}$$

If n be the angular velocity of the earth's rotation

$$u' = r\dot{\theta}', \quad v' = r(\dot{\phi}' + n)\sin\theta', \quad w' = \dot{r},$$

$$\theta_1 = -(\dot{\phi}' + n)\sin\theta', \quad \theta_2 = \dot{\theta}', \quad \theta_3 = (\dot{\phi}' + n)\cos\theta',$$

and the preceding expressions for the component accelerations become

$$\frac{d}{dt}(r\dot{\theta}') - r(\dot{\phi}' + n)^2 \sin\theta' \cos\theta' + \dot{r}\dot{\theta}',$$

$$\frac{d}{dt}\{r(\dot{\phi}' + n)\sin\theta'\} + \dot{r}(\dot{\phi}' + n)\sin\theta' + r\dot{\theta}'(\dot{\phi}' + n)\cos\theta',$$

$$\ddot{r} - r\dot{\theta}^2 - r(\dot{\phi}' + n)^2 \sin^2\theta'.$$

But $\theta' = \theta + u$, $\phi' = \phi + v$, where θ, ϕ are the co-latitude and longitude of the element in its undisturbed position; also since the vertical motion is slow in comparison with the horizontal motion (since the oscillations are long waves), we may neglect $\ddot{r}$, $\dot{r}$, and the above expressions become

$$\begin{array}{rl} -r(n^2 + 2n\dot{v})\sin^2\theta & \text{in the direction of } r, \\ r\ddot{u} - r(n^2 + 2n\dot{v})\sin\theta\cos\theta & \quad ,, \qquad ,, \qquad ,, \;\theta, \\ r\ddot{v}\sin\theta + 2rn\dot{u}\cos\theta & \quad ,, \qquad ,, \qquad ,, \;\phi. \end{array}$$

Hence the equations of motion are

$$\left.\begin{aligned}\frac{1}{\rho}\frac{dp}{dr} &= \frac{dV}{dr} + r\,(n^2 + 2n\dot{v})\sin^2\theta \\ \frac{1}{\rho}\frac{dp}{d\theta} &= \frac{dV}{d\theta} - r^2\ddot{u} + r^2\,(n^2 + 2n\dot{v})\sin\theta\cos\theta \\ \frac{1}{\rho}\frac{dp}{d\phi} &= \frac{dV}{d\phi} - r^2\ddot{v}\sin^2\theta - 2r^2n\dot{u}\sin\theta\cos\theta\end{aligned}\right\}\dots\dots\dots\dots(8),$$

where V is the attraction potential of the forces.

According to the theory of long waves explained in Chapter XVII., the pressure is assumed to be equal to the hydrostatic pressure which would exist if liquid had no motion, and were under the action of forces which would preserve the form of its free surface unaltered. It therefore follows from § 369, or directly from (8) by putting $\dot{u}$, $\ddot{u}$, $\dot{v}$, $\ddot{v}$ equal to zero and multiplying by dr, $d\theta$, $d\phi$ and integrating, that

$$p/\rho = V' + \tfrac{1}{2}n^2r^2\sin^2\theta + \text{const.},$$

where V' is the potential of the fictitious forces which would produce an equilibrium tide of height $\mathfrak{h}$. By (1) and (2) it follows that the variable part of this potential is $g\mathfrak{h}r^2/a^2 + E/r$, whence

$$p/\rho = g\mathfrak{h}r^2/a^2 + E/r + \tfrac{1}{2}n^2r^2\sin^2\theta + \text{const.}$$

Now V is the potential of a system of forces which would produce an equilibrium tide of height $\mathfrak{e}$, whence

$$V = g\mathfrak{e}r^2/a^2 + E/r + \text{const.},$$

and therefore

$$p/\rho - V = g\,(\mathfrak{h} - \mathfrak{e})\,r^2/a^2 + \tfrac{1}{2}n^2r^2\sin^2\theta + \text{const.}$$

Substituting this value of $p/\rho - V$ in the last two of (8) and putting $au = \xi$, $av = \eta$, so that ξ, $\eta\sin\theta$ are the co-latitudinal and longitudinal displacements, and remembering that in the small terms we may put $r = a$, we obtain

$$\left.\begin{aligned}\ddot{\xi} - 2n\dot{\eta}\sin\theta\cos\theta &= -\frac{g}{a}\frac{d}{d\theta}(\mathfrak{h} - \mathfrak{e}) \\ \ddot{\eta}\sin\theta + 2n\dot{\xi}\cos\theta &= -\frac{g}{a\sin\theta}\frac{d}{d\phi}(\mathfrak{h} - \mathfrak{e})\end{aligned}\right\}\dots\dots(9),$$

and (7) becomes

$$\mathfrak{h}a\sin\theta + \frac{d}{d\theta}(\gamma\xi\sin\theta) + \frac{d}{d\phi}(\gamma\eta\sin\theta) = 0\dots\dots(10).$$

The solution of these equations determines the oscillations of the ocean.

438. The value of $\mathfrak{e}$ is given by (4), and from this value it appears that $\mathfrak{e}$ consists of three sets of terms. The first set which in the equilibrium theory gives rise to tides of long period vary very slowly, and it is known from physical astronomy that these terms can be expanded in a series of terms of the type $A\cos(2nft+\alpha)$; the second and third set respectively involve ϕ and 2ϕ; it therefore follows that $\mathfrak{e}$ can be expanded in a series of terms of the type $e\cos(2nft+k\phi+\alpha)$, where e is a function of the co-latitude alone and of the elements of the orbit of the disturbing body.

It also follows from (4) that the tides of long period do not depend on the longitude, hence $k=0$,

$$\left.\begin{aligned} e &= E\left(\tfrac{1}{3}-\cos^2\theta\right) \\ \mathfrak{e} &= E\left(\tfrac{1}{3}-\cos^2\theta\right)\cos(2nft+\alpha) \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(11).$$

In the lunar fortnightly tide $f=\frac{1}{28}$ approximately.

In the diurnal tides $f=\frac{1}{2}$ approximately, $k=1$,

$$e = E\sin\theta\cos\theta$$

$$\mathfrak{e} = E\sin\theta\cos\theta\cos(nt+\phi+\alpha)\quad\ldots\ldots\ldots\ldots(12).$$

In the semi-diurnal tides, $f=1$ approximately, $k=2$,

$$e = E\sin^2\theta$$

$$\mathfrak{e} = E\sin^2\theta\cos(2nt+2\phi+\alpha)\ldots\ldots\ldots\ldots\ldots(13).$$

In each of the three preceding equations, the quantity E is a function of the elements of the orbit of the disturbing body.

We shall therefore assume that,

$$\left.\begin{aligned} \mathfrak{e} &= e\cos(2nft+k\phi+\alpha) \\ \mathfrak{h} &= h\cos(2nft+k\phi+\alpha) \\ \xi &= x\cos(2nft+k\phi+\alpha) \\ \eta &= y\sin(2nft+k\phi+\alpha) \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

where e, h, x, y are functions of the co-latitude alone; and we shall also suppose that γ is a function of the co-latitude alone.

Let $\qquad m=n^2a/g, \quad u=h-e.$

Substituting from (14) in (9) and (10) we obtain

$$\left.\begin{aligned} xf^2+yf\sin\theta\cos\theta &= \frac{1}{4m}\frac{du}{d\theta} \\ yf^2\sin\theta+xf\cos\theta &= -\frac{ku}{4m\sin\theta} \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(15),$$

$$ha + k\gamma y + \text{cosec}\,\theta \frac{d}{d\theta}(\gamma x \sin\theta) = 0 \;\ldots\ldots\ldots\ldots(16).$$

Solving (16) for x and y, we obtain

$$\left.\begin{aligned} 4mx\,(f^2 - \cos^2\theta) &= \frac{du}{d\theta} + \frac{ku}{f}\cot\theta \\ 4my\sin\theta\,(f^2 - \cos^2\theta) &= -\frac{\cos\theta}{f}\frac{du}{d\theta} - \frac{ku}{\sin\theta} \end{aligned}\right\} \;\ldots\ldots\ldots\ldots(17).$$

Whence substituting the preceding values of x, y and u in (17), we obtain

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\frac{\gamma\,(\sin\theta\, du/d\theta + kf^{-1}u\cos\theta)}{f^2 - \cos^2\theta}\right\}$$
$$- \frac{k\gamma\,(f^{-1}\cos\theta\, du/d\theta + ku\,\text{cosec}\,\theta)}{\sin\theta\,(f^2 - \cos^2\theta)},$$
$$+ 4ma\,(u + e) = 0 \;\ldots\ldots\ldots\ldots(18).$$

This is equivalent to Laplace's equation[1] for determining the tidal oscillations of an ocean, whose depth γ is a function of the latitude alone.

Tides of Long Period.

439. Laplace in considering these tides does not employ (18), but endeavours to show that on account of the friction of the ocean against its bed, the values of these tides will be the same as the corresponding values furnished by the equilibrium theory; and he assumes that the effect of this friction upon any element of the liquid, can be represented by a force proportional to the velocity of that element. One objection to this hypothesis is that it is in complete disagreement with the theory of the motion of a viscous liquid, which, as we shall see in the next chapter, shows that the effect of friction is to introduce terms of the form $\nu\nabla^2 u$, $\nu\nabla^2 v$, $\nu\nabla^2 w$ into the general equations of motion, where ν is a constant depending on the viscosity of the liquid[2].

Another objection, which has been urged by Prof. Darwin, is as follows[3]. "In systems where resistances are proportional to the

[1] *Mécanique Céleste*, Livre IV. § 3, (4).

[2] The problem of Waves in a slightly viscous liquid will be considered in Chapter XXIII.

[3] *Proc. Roy. Soc.* 1886.

velocity, it is usual to specify the resistance by a modulus of decay, viz., that period in which a velocity is reduced by friction to e^{-1} or $1 \div 2{\cdot}783$ of its initial value, and the friction contemplated by Laplace is such that the modulus of decay is short compared with the semi-period of oscillation. The quickest of the tides of long period is the fortnightly tide, hence for the applicability of Laplace's conclusion, the modulus of decay must be short compared with a week. Now it seems practically certain that the friction of the ocean bed would not materially affect the velocity of a slow ocean current in a day or two. Hence we cannot accept Laplace's hypothesis as to the effect of friction."

Laplace's argument is as follows. He supposes that the co-latitudinal and longitudinal components of the resistance are represented by the terms $\mathfrak{E}\dot{\xi}$, $\dot{\eta}\mathfrak{E}\sin\theta$. Now the terms $\ddot{\xi}$, $\ddot{\eta}$ depend upon f^2, and may be neglected if f is small; hence (9) and (10) become

$$\mathfrak{E}\dot{\xi} - 2n\dot{\eta}\sin\theta\cos\theta = -\frac{g}{a}\frac{d}{d\theta}(\mathfrak{h} - \mathfrak{e}),$$

$$\mathfrak{E}\dot{\eta}\sin\theta + 2n\dot{\xi}\cos\theta = 0,$$

$$\mathfrak{h}a\sin\theta + \frac{d}{d\theta}(\gamma\xi\sin\theta) = 0,$$

since none of the quantities depend upon ϕ. From the first two we obtain

$$\left(\mathfrak{E} + \frac{4n^2}{\mathfrak{E}}\cos^2\theta\right)\dot{\xi} = -\frac{g}{a}\frac{d}{d\theta}(\mathfrak{h} - \mathfrak{e}).$$

Substituting in this the value of ξ from (14) we obtain

$$2nf\left(\mathfrak{E} + \frac{4n^2}{\mathfrak{E}}\cos^2\theta\right)x = -\frac{g}{a}\frac{d}{d\theta}(\mathfrak{h} - \mathfrak{e}),$$

whence if f is small compared with $\mathfrak{E}$, the left-hand side may be neglected and we obtain $\mathfrak{h} - \mathfrak{e} = 0$.

440. We shall now give Prof. Darwin's solution of this problem[1].

It is assumed that the ocean is of uniform depth; hence putting $\beta = 4ma/\gamma$, $\mu = \cos\theta$; and remembering that $k = 0$, $e = E(\frac{1}{3} - \mu^2)$, (18) becomes

$$\frac{d}{d\mu}\left\{\frac{1-\mu^2}{\mu^2 - f^2}\frac{du}{d\mu}\right\} = \beta\left\{u + E\left(\tfrac{1}{3} - \mu^2\right)\right\} \quad\ldots\ldots\ldots\ldots(19).$$

[1] *Proc. Roy. Soc.* 1886.

The symmetry of the motion requires that u should be an even function of μ, hence at the equator where $\mu=0$, we must have $du/d\mu=0$.

Let us assume

$$\frac{1}{\mu^2-f^2}\frac{du}{d\mu}=B_1\mu+B_3\mu^3+\ldots B_{2n+1}\mu^{2n+1}+\ldots.$$

Then

$$\frac{1-\mu^2}{\mu^2-f^2}\frac{du}{d\mu}=B_1\mu+(B_3-B_1)\mu^3+\ldots(B_{2n+1}-B_{2n-1})\mu^{2n+1}+\ldots,$$

$$\frac{d}{d\mu}\left(\frac{1-\mu^2}{\mu^2-f^2}\frac{du}{d\mu}\right)=B_1+3(B_3-B_1)\mu^2\ldots$$
$$+\ldots(2n+1)(B_{2n+1}-B_{2n-1})\mu^{2n}+\ldots\ldots\ldots(20).$$

Again

$$\frac{du}{d\mu}=-B_1f^2\mu+(B_1-f^2B_3)\mu^3+\ldots(B_{2n-1}-f^2B_{2n+1})\mu^{2n+1}+\ldots,$$

whence $$u=C-\tfrac{1}{2}f^2B_1\mu^2+\tfrac{1}{4}(B_1-f^2B_3)\mu^4+\ldots\ldots$$
$$+\frac{1}{2n}(B_{2n-3}-f^2B_{2n-1})\mu^{2n}+\ldots\ldots\ldots\ldots(21),$$

where C is a constant. Substituting from (20) and (21) in (19), and equating coefficients we obtain

$$\left.\begin{array}{r} C=-\tfrac{1}{3}E+B_1/\beta \\ B_3-B_1\left(1-\dfrac{f^2\beta}{2.3}\right)+\tfrac{1}{3}\beta E=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ B_{2n+1}-B_{2n-1}\left\{1-\dfrac{f^2\beta}{2n(2n+1)}\right\}-\dfrac{\beta B_{2n-3}}{2n(2n+1)}=0 \end{array}\right\}\ldots\ldots\ldots\ldots(22).$$

Hence all the constants C, B_3, $B_5\ldots$ are expressible in terms of B_1, which is apparently indeterminate.

If we put

$$-\tfrac{1}{6}\beta B_{-1}=\tfrac{1}{3}\beta E,\text{ or } B_{-1}=-2E,$$

the last of equations (22) will hold for all values of n from 1 to ∞. Writing this equation in the form

$$\frac{B_{2n+1}}{B_{2n-1}}=1-\frac{f^2\beta}{2n(2n+1)}+\frac{\beta}{2n(2n+1)}\frac{B_{2n-3}}{B_{2n-1}},$$

we see at once that when n is large B_{2n+1}/B_{2n-1} either tends to become infinitely small or it does not.

If this ratio does not become infinitely small, it follows that the successive B's tend to become equal to one another, and so also do the coefficients $B_{2n-1} - f^2 B_{2n+1}$ in the expression for $du/d\mu$. We may therefore put

$$\frac{du}{d\mu} = L + \frac{M}{1-\mu^2},$$

where L and M are finite quantities which do not vanish for any value of μ, hence

$$\frac{du}{d\theta} = -L(1-\mu^2)^{\frac{1}{2}} - \frac{M}{(1-\mu^2)^{\frac{1}{2}}}.$$

Substituting this value of $du/d\theta$ in the first of (17), and putting $k=0$, it follows that at the pole where $\mu = 1$, x and therefore ξ are infinite. Hence the hypothesis that B_{2n+1}/B_{2n-1} does not tend to become infinitely small, makes the velocity infinite at either pole, which is obviously contrary to the facts of the case; and we therefore conclude that B_{2n+1}/B_{2n-1} does tend to become infinitely small.

Writing the last of (22) in the form

$$\frac{B_{2n-1}}{B_{2n-3}} = \frac{-\dfrac{\beta}{2n(2n+1)}}{1 - \dfrac{f^2\beta}{2n(2n+1)} - \dfrac{B_{2n+1}}{B_{2n-1}}} \quad \text{..............(23)},$$

it follows that this ratio may be written in the form of the continued fraction

$$\frac{B_{2n-1}}{B_{2n-3}} = \frac{-\dfrac{\beta}{2n(2n+1)}}{1 - \dfrac{f^2\beta}{2n(2n+1)}} \,\Bigg|\, + \frac{\dfrac{\beta}{(2n+2)(2n+3)}}{1 - \dfrac{f^2\beta}{(2n+2)(2n+3)}} \,\Bigg|\, + \; \&c.$$

This continued fraction gives the value which this ratio must have when the water covers the whole globe.

Let N_n denote the value of the continued fraction, then remembering that $B_{-1} = -2E$, we have

$$B_1 = 2EN_1,\; B_3 = -N_2 B_1 = -2EN_1N_2,$$

$$B_5 = -2EN_1N_2N_3 \;\&c.,$$

$$C = -\tfrac{1}{3}E + 2EN_1/\beta.$$

We therefore obtain

$$h = u + E(\tfrac{1}{3} - \mu^2)$$
$$= C + \tfrac{1}{3}E - (E + \tfrac{1}{2}f^2B_1)\mu^2 + \tfrac{1}{4}(B_1 - f^2B_3)\mu^4 + \ldots,$$

or $$h/E = 2N_1/\beta - (1 + f^2N_1)\mu^2 + \tfrac{1}{2}N_1(1 + f^2N_2)\mu^4$$
$$- \tfrac{1}{3}N_1N_2(1 + f^2N_3)\mu^6 + \ldots\ldots\ldots\ldots\ldots\ldots \quad (24).$$

The height $\mathfrak{h}$ of the tide is equal to $h\cos(2nft + \alpha)$ and the height of the equilibrium tide is $\mathfrak{e} = E(\frac{1}{3} - \mu^2)\cos(2nft + \alpha)$.

In the paper from which this investigation is taken, Prof. Darwin has made some numerical calculations for determining the values of the fortnightly tide when the depth of the ocean is 3000 and 1200 fathoms respectively; and he finds that in the case of the oceans upon the earth, this tide is smaller than half its equilibrium value, but with a deeper ocean the tide would approximate towards its equilibrium value.

The Diurnal Tides.

441. In these tides $k = 1$, $f = \frac{1}{2}$, $e = E\sin\theta\cos\theta$, also $\gamma = l(1 - q\cos^2\theta)$.

In order to solve (18) let us assume

$$u = F_0 + F_1\left(\frac{2lq}{ma}\right) + F_2\left(\frac{2lq}{ma}\right)^2 + \ldots\ldots$$

where the F's are functions of θ but not of l; substituting in (18) and equating coefficients of powers of l, we at once obtain

$$F_0 = -e = -E\sin\theta\cos\theta.$$

To determine F_1, put $u = F_0$ in the left-hand side of (18), then

$$\frac{\gamma(\sin\theta\, du/d\theta + 2u\cos\theta)}{\frac{1}{4} - \cos^2\theta} = -\frac{4E\gamma\{\sin\theta(2\cos^2\theta - 1) + 2\sin\theta\cos^2\theta\}}{1 - 4\cos^2\theta}$$
$$= 4E\gamma\sin\theta,$$

also

$$\frac{\gamma(2\cos\theta\, du/d\theta + u\operatorname{cosec}\theta)}{\sin\theta(\frac{1}{4} - \cos^2\theta)} = -\frac{4E\gamma\{2\cos\theta(2\cos^2\theta - 1) + \cos\theta\}}{\sin\theta(1 - 4\cos^2\theta)}$$
$$= 4E\gamma\cot\theta,$$

whence the first two terms of (18) are equal to

$$8Elq\sin\theta\cos\theta = -8lqF_0.$$

We thus obtain $$8lqF_0' = 8lqF_1,$$

therefore $$F_0 = F_1.$$

Proceeding in the same way it can be shown that

$$F_n = F_{n-1} = \ldots\ldots = F_0,$$

whence the value of u finally becomes

$$u = F_0 \left\{1 + \frac{2lq}{ma} + \left(\frac{2lq}{ma}\right)^2 + \ldots\ldots\right\}$$

$$= -\frac{e}{1 - 2lq/ma}.$$

Whence $$h = e + u = -\frac{2lqe/ma}{1 - 2lq/ma} \qquad \text{.....................(25)}.$$

The peculiarity of this tide is, that when $q = 0$ so that the depth of the ocean is everywhere uniform, the tide vanishes.

If q is not zero and E is positive, e will be positive if the place of observation is in north latitude. If therefore the ocean is shallower at the poles than at the equator, q is positive and therefore when the disturbing body is in the meridian of the place of observation h is negative, and the tide is inverted.

442. The evanescence of this tide applies only to the elevation of the water; the velocity of the latter which depends on ξ and η does not vanish. Putting $u = -e$ we obtain from (17),

$$x = E/m, \quad y \sin\theta = -Em^{-1}\cos\theta.$$

Confining our attention to a single disturbing body, it appears from (4) that $E = 3Ma^2 \sin\delta \cos\delta/D^3 g$; hence if the declination of the disturbing body is north, E is positive, and therefore in north latitude the motion of the water is from north to south, and the longitudinal velocity vanishes at the equator.

The Semi-diurnal Tides.

443. We shall only be able to solve the problem of the semi-diurnal tides when $q = 1$ and $q = 0$. In the former case $\gamma = l\sin^2\theta$, and the height of the tide can be found by a similar method to that employed in the preceding section.

We have $k=2, f=1, e=E\sin^2\theta$. Let us therefore endeavour to find a solution of (18) of the form

$$u = F_0 + F_1\left(\frac{2l}{ma}\right) + F_2\left(\frac{2l}{ma}\right)^2 + \ldots\ldots$$

where the F's are functions of θ but not of l. We obtain as before

$$F_0 = -e = -E\sin^2\theta.$$

To determine F_1 put $u=F_0$ in the left-hand side of (18), then

$$\frac{\gamma(\sin\theta\, du/d\theta + 2u\cos\theta)}{1-\cos^2\theta} = -4El\sin^2\theta\cos\theta$$

$$\frac{2\gamma(\cos\theta\, du/d\theta + 2u\operatorname{cosec}\theta)}{\sin\theta(1-\cos^2\theta)} = -4El(1+\cos^2\theta).$$

Whence the first two terms of (18) are equal to

$$8El\sin^2\theta = -8lF_0.$$

We thus obtain $$F_1 = F_0.$$

Proceeding in the same way we obtain

$$F_n = F_{n-1} = \ldots\ldots = F_0,$$

and the value of u finally becomes

$$u = F_0\left\{1 + \frac{2l}{ma} + \left(\frac{2l}{ma}\right)^2 + \ldots\ldots\right\}$$

$$= -\frac{e}{1-2l/ma},$$

whence $$h = e + u = -\frac{2le/ma}{1-2l/ma} \ldots\ldots\ldots\ldots\ldots\ldots(26).$$

If $l/ma < \frac{1}{2}$, it appears that when the disturbing body is on the meridian, the tide is inverted.

444. Laplace has also solved the equation determining these tides, when the ocean is of uniform depth, which leads to a solution involving a continued fraction similar to that of § 440.

Let $q=0$, $\beta = 4ma/l$, $\nu = \sin\theta$, so that $e=E\nu^2$. Changing the variable from θ to ν, (18) becomes

$$\nu^2(1-\nu^2)\frac{d^2u}{d\nu^2} - \nu\frac{du}{d\nu} - (8 - 2\nu^2 - \beta\nu^4)u + E\beta\nu^6 = 0 \ldots(27).$$

In order to satisfy this equation, let us assume

$$u = B_0 + (B_2 - E)\nu^2 + B_4\nu^4 + B_6\nu^6 + \ldots\ldots + B_{2n}\nu^{2n} \ldots\ldots(28).$$

Substituting the preceding series in (27) and equating coefficients we obtain

$$B_0 = 0, \ B_2 = E,$$

and $$2n(2n+6) B_{2n+4} - 2n(2n+3) B_{2n+2} + \beta B_{2n} = 0 \ldots\ldots (29).$$

By means of this equation the values of the B's can be determined in terms of B_2, B_4. Now $B_2 = E$; also since the motion is symmetrical with respect to the equator it follows from (17) that x and therefore $du/d\theta$ must vanish at the equator; hence B_4 must be determined from the condition that $du/d\theta = 0$ when $\theta = \frac{1}{2}\pi$. Writing (29) in the form,

$$\frac{B_{2n+4}}{B_{2n+2}} = \frac{2n+3}{2n+6} - \frac{\beta}{2n(2n+6)} \frac{B_{2n}}{B_{2n+2}},$$

it follows that in order that the series should be convergent, it is necessary that B_{2n+2}/B_{2n} should tend to a limit < 1.

Now this quantity tends to become infinitely small or it does not; in the latter case

$$\frac{B_{2n+4} \nu^{2n+4}}{B_{2n+2} \nu^{2n+2}} = \frac{2n+3}{2n+6} \nu^2 = \left(1 - \frac{3}{2n}\right) \nu^2,$$

ultimately when n is very large.

Now this is the degree of ultimate convergence of the series for $(1-\nu^2)^{\frac{1}{2}}$, hence the series for u is convergent and we may therefore put

$$u = A + B(1-\nu^2)^{\frac{1}{2}},$$

where A and B are finite for all values of ν.

Differentiating (28) with respect to ν, the convergence of the series for $du/d\nu$ depends upon the value of

$$(2n+4) B_{2n+4} \nu^2 / (2n+2) B_{2n+2}.$$

Now by (29)

$$\frac{(2n+4) B_{2n+4}}{(2n+2) B_{2n+2}} = \frac{(2n+4)(2n+3)}{(2n+2)(2n+6)} \nu^2$$

$$= \left(1 - \frac{1}{2n}\right) \nu^2,$$

when n is very large. Now this is the degree of convergence of the series for $(1-\nu^2)^{-\frac{1}{2}}$; we may therefore put

$$\frac{du}{d\nu} = C + D(1-\nu^2)^{-\frac{1}{2}},$$

where C and D are finite for all values of ν; also since

$$\frac{du}{d\theta} = (1-\nu^2)^{\frac{1}{2}}\frac{du}{d\nu} = C(1-\nu^2)^{\frac{1}{2}} + D,$$

it follows that at the equator where $\nu = 1$, $du/d\theta = D$. Hence the hypothesis that B_{2n+4}/B_{2n+2} *does not* tend to become infinitely small, makes $d\xi/dt$ finite at the equator; we therefore conclude that this ratio *does* tend to become infinitely small as n indefinitely increases.

Writing (29) in the form

$$\frac{B_{2n+2}}{B_{2n}} = \frac{\frac{1}{2}\beta}{2n^2+3n-(2n^2+6n)\,B_{2n+4}/B_{2n+2}},$$

it follows that B_{2n+2}/B_{2n} is expressible in the form of the continued fraction

$$\frac{B_{2n+2}}{B_{2n}} = \frac{\frac{1}{2}\beta}{2n^2+3n}\,\overline{\Big|}\,\frac{-(n^2+3n)\beta}{2(n+1)^2+3(n+1)}\,\overline{\Big|}\,\frac{-\{(n+1)^2+3(n+1)^2\}\beta}{2(n+2)^2+3(n+2)}\,\overline{\Big|}-\&c.$$

Putting N_n for the continued fraction, we obtain

$$B_2 = E,\; B_4 = EN_1,\; B_6 = EN_1N_2 \;\&c.$$

The solution of the problem of the semi-diurnal tides in an ocean of uniform depth by means of this continued fraction was given by Laplace without explanation; it was attacked by Airy[1], and by Ferrel[2], but was justified by Sir W. Thomson[3] and the process was worked out and explained by Prof. Darwin[4] as above.[5]

445. The following numerical results are given by Laplace. The quantity m is the ratio of the centrifugal force to gravity at the equator, and is equal to $\frac{1}{289}$; if therefore we put β successively

[1] "Tides and Waves," *Encyc. Met.*

[2] "Tidal Researches," *U. S. Coast Survey.*

[3] *Phil. Mag.* 1875.

[4] "Tides," *Encyc. Brit.*

[5] The reasoning which lies at the bottom of the investigations of §§ 440 and 444, may I think be rendered clearer by the following considerations.

Let us suppose that we have to find the value of a function which satisfies (i) a given differential equation, (ii) certain other conditions. Then if we seek for a solution in the form of a *series*, and determine all the coefficients so as to satisfy (i) and (ii), the series will not be the solution we require *unless it be convergent.* Similarly if the conditions (i) and (ii) enable us to determine all the coefficients in terms of a single unknown quantity A, it does not follow that A is indeterminate; for if by assigning any particular value to A, the resulting series could be made *divergent*, this value would have to be excluded. The quantity A is therefore not really indeterminate, but must be found from the condition that the series should be *convergent.*

equal to 40, 10, and 5, the corresponding depths of the ocean will be $\frac{1}{2890}$, $\frac{1}{722\cdot5}$, $\frac{1}{361\cdot25}$ of the earth's radius. Also since $h = E\nu^2 + u$, Laplace finds the following values of h in the three cases, viz.,

$$\beta = 40$$

$$h = E(\nu^2 + 20\cdot1862\,\nu^4 + 10\cdot1164\,\nu^6 - 13\cdot1047\,\nu^8 - 15\cdot4488\,\nu^{10}$$
$$- 7\cdot4581\nu^{12} - 2\cdot1975\nu^{14} - \cdot4501\,\nu^{16} - \cdot0687\,\nu^{18}$$
$$- \cdot0082\,\nu^{20} - \cdot0008\,\nu^{22} - \cdot0001\,\nu^{24}),$$

$$\beta = 10$$

$$h = E(\nu^2 + 6\cdot1960\,\nu^4 + 3\cdot2474\,\nu^6 + \cdot7238\,\nu^8 + \cdot0919\,\nu^{10}$$
$$+ \cdot0076\,\nu^{12} + \cdot0004\,\nu^{14}),$$

$$\beta = 5$$

$$h = E(\nu^2 + \cdot7504\,\nu^4 + \cdot1566\,\nu^6 + \cdot0157\,\nu^8 + \cdot0009\,\nu^{10}).$$

From these equations we see that h vanishes when $\nu = 0$, hence there is no tide at either pole.

At the equator $\nu = 1$, and we find

$$\beta = 40, \quad h = -7\cdot4344\,E$$
$$\beta = 10, \quad h = 11\cdot2671\,E$$
$$\beta = 5, \quad h = 1\cdot9236\,E.$$

When $\beta = 40$, h is negative which shows that at the equator the tide is inverted; but in the neighbourhood of the poles where ν is small, the tides are direct; hence there is a certain latitude, which is approximately 18^0 in which the tide vanishes, and which is therefore a nodal line of evanescent tide. In the other two cases the tides are always direct; hence it follows that if the depth of the ocean is $\frac{1}{2890}$ ths of the earth's radius, or 1200 fathoms, the tides will vanish in latitude 18^0, and in lower latitudes will be inverted; as the depth of the ocean increases the latitude of the evanescent tide increases until it ultimately coincides with the equator, and for greater depths the tides are direct everywhere. This critical depth lies between 1200 and 4800 fathoms.

Free Oscillations of an Ocean of Uniform Depth.

446. Before passing on to consider the canal theory of tides, we shall consider the problem of the *free* oscillations of an ocean of uniform depth which completely envelopes a sphere[1].

Let a be the radius of the sphere, h the depth of the ocean when undisturbed. Let the equation of the surface of the sea be

$$r = a + h + \Sigma_1^\infty Y_n \quad\text{.......................(30)},$$

where Y_n is a spherical surface harmonic, and the $2n + 1$ constants which it contains are functions of the time. Since $\iint Y_n dS$ vanishes when the integration is taken over the surface of a sphere, the condition of constancy of volume is satisfied by (30).

Since $d\phi/dr$ vanishes when $r = a$, the velocity potential ϕ must be of the form

$$\phi = \Sigma \{(n+1)(r/a)^n + n(a/r)^{n+1}\} Z_n \quad\text{...........(31)},$$

where Z_n is another spherical surface harmonic.

The condition that (30) should be a bounding surface is

$$\Sigma \frac{dY_n}{dt} - \frac{d\phi}{dr} = 0,$$

when $r = a + h$; whence writing b for $a + h$, we obtain

$$\frac{dY_n}{dt} = a^{-1} n(n+1) \{(b/a)^{n-1} - (a/b)^{n+2}\} Z_n \quad\text{.........(32)}.$$

The equation determining the pressure is

$$\frac{p}{\rho} - V + \frac{d\phi}{dt} = \text{const.} \quad\text{.....................(33)},$$

where V is the attraction potential, and the square of the velocity is as usual neglected. By § 371, the value of V at the surface is

$$V = \frac{E}{r} + 4\pi\rho b \Sigma \frac{Y_n}{2n+1},$$

$$= \frac{E}{b} - \Sigma \left(\frac{E}{b^2} - \frac{4\pi\rho b}{2n+1}\right) Y_n,$$

[1] Lamb, *Motion of Fluids*, p. 197. Thomson, *Phil. Trans.* 1863, p. 608.

where E is the mass of the sphere and liquid, and ρ is the density of the latter. If σ be the density of the sphere

$$E = \tfrac{4}{3}\pi a^3 \sigma + \tfrac{4}{3}\pi\rho\,(b^3 - a^3).$$

Whence (33) becomes

$$\frac{p}{\rho} - \frac{E}{b} + \Sigma\left(\frac{E}{b^2} - \frac{4\pi\rho b}{2n+1}\right) Y_n$$
$$+ \Sigma\,\{(n+1)\,(b/a)^n + n\,(a/b)^{n+1}\}\,\frac{dZ_n}{dt} = \text{const.}$$

At the free surface $p = \text{const.}$; whence putting $E/b^2 = g$, we obtain

$$-\{(n+1)\,(b/a)^n + n\,(a/b)^{n+1}\}\,\frac{dZ_n}{dt} = \{g - 4\pi\rho b/(2n+1)\}\,Y_n$$
$$= g\left[1 - \frac{3\rho b^3}{(2n+1)\,\{\sigma a^3 + \rho\,(b^3 - a^3)\}}\right] Y_n \ldots\ldots (34).$$

Eliminating Z_n between (32) and (34) we obtain

$$\frac{d^2 Y_n}{dt^2} + \frac{4\pi^2}{T_n} Y_n = 0,$$

where T_n, the period of oscillation, is determined by the equation

$$T_n^2 = 4\pi^2 a g^{-1}\,\{(n+1)\,(b/a)^n + n\,(a/b)^{n+1}\}$$
$$\div\, n\,(n+1)\,\{(b/a)^{n-1} - (a/b)^{n+2}\}\left[1 - \frac{3\rho b^3}{(2n+1)\,\{\sigma a^3 + \rho(b^3 - a^3)\}}\right] \ldots (35).$$

If $\sigma < \rho$ the value of T_1 will be imaginary, and the motion is unstable. If therefore a spherical nucleus is surrounded by liquid of uniform depth, the equilibrium will be unstable if the density of the liquid is greater than that of the nucleus, and the nucleus will float on the liquid with a portion of its surface protruding.

If $a = 0$, (35) becomes

$$T_n^2 = \frac{2\pi^2\,(2n+1)\,b}{n\,(n-1)\,g};$$

which determines the period of oscillation of a spherical mass of liquid under the influence of its own attraction.

If h be small compared with a, (35) becomes

$$T_n^2 = 4\pi^2 a^2/n\,(n+1)\,\{1 - 3\rho/(2n+1)\,\sigma\}\,gh;$$

a result due to Laplace.

The Canal Theory of Tides[1].

447. The defect of Laplace's theory when applied to the tides as they actually exist in the oceans covering the earth, consists in the circumstance that this theory is based upon the assumption that the whole earth is covered with water; whereas the existence of large continents must seriously affect the accuracy of results deduced from this theory. Another theory has been developed by Airy, which is usually known as the canal theory of tides, whose object is to investigate the tidal motion of water due to the disturbing influence of the sun and moon, in a narrow canal whose form is that of a small circle upon the earth.

448. Since the lateral dimensions of the canal are supposed to be small in comparison with the radius of the earth, the problem may be treated as one of two-dimensional motion. Let the origin be taken in the bottom of the canal, and let the axis of x be measured along the canal, and that of y vertically upwards. Let ξ be the displacement in the direction of x, of an element of liquid whose undisturbed co-ordinates are (x, y); X, Y the component forces parallel to the axes due to the disturbing body; h the depth of the canal, η the height of the tide.

The equations of motion are

$$\ddot{\xi} = X - \frac{1}{\rho}\frac{dp}{dx} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36),$$

$$\ddot{\eta} = Y - g - \frac{1}{\rho}\frac{dp}{dy} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (37).$$

Since the vertical acceleration is small compared with the horizontal acceleration, $\ddot{\eta}$ may be neglected; also since the disturbing force is small compared with the attraction of the earth, the pressure at a given depth will be approximately equal to the hydrostatic pressure due to the height of the free surface; we may therefore put

$$p = g\rho\,(h + \eta - y).$$

Substituting in (36) we obtain

$$\ddot{\xi} = X - g\frac{d\eta}{dx}.$$

[1] Airy, "Tides and Waves," Sec. VI. *Encyc. Met.*

By § 403, the equation of continuity is

$$\eta/h = -d\xi/dx \qquad (38),$$

whence
$$\ddot{\xi} = X + gh\frac{d^2\xi}{dx^2} \qquad (39).$$

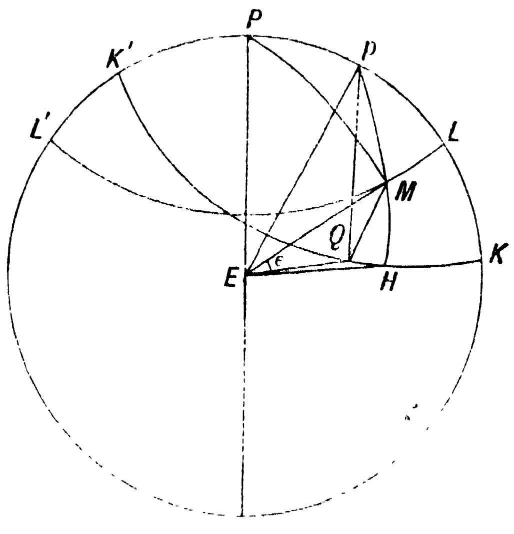
In the following applications, X will be of the form $A\sin(nt - mx + \alpha)$, where A and α are constants; substituting this value of X in (39), and integrating we obtain

$$\left.\begin{aligned} \xi &= \frac{A}{m^2gh - n^2}\sin(nt - mx + \alpha) \\ \eta &= \frac{Ahm}{m^2gh - n^2}\cos(nt - mx + \alpha) \end{aligned}\right\} \qquad (40).$$

whence

This is the portion of ξ which depends upon the disturbing body, and therefore constitutes the forced oscillation. The free oscillations are represented by the complimentary function which is obtained by integrating (39) with $X = 0$.

449. We shall now suppose the disturbing body to be the moon, which is assumed to revolve with angular velocity n, in an orbit whose projection upon the earth is a small circle, and that the canal is any other small circle upon the earth.



In the figure let P be the pole of the earth, M the projection of the moon, LL' the small circle described by it round the pole; let KK' be the canal, Q any point on it. Let $MQ = \epsilon$, $pQ = \alpha$, $pM = \beta$, $Pp = \gamma$; also let the angle $KpQ = \phi$, $HpQ = \theta$; also let $LPM = nt$, $PM = \frac{1}{2}\pi - \delta$, so that δ is the declination of the moon.

From the spherical triangle QpM we obtain

$$\cos\epsilon = \cos\theta \sin\alpha \sin\beta + \cos\alpha \cos\beta \qquad (41).$$

By § 429, the potential of the forces acting on an element at Q is

$$V = \frac{M}{D} + \frac{E}{r} + A + \frac{Mr^2}{2D^3}(3\cos^2\epsilon - 1),$$

or,

$$V = \frac{M}{D} + \frac{E}{r} + A + \frac{Mr^2}{2D^3}\{3(\cos\theta \sin\alpha \sin\beta + \cos\alpha \cos\beta)^2 - 1\},$$

by (41). The force at Q along QK' is

$$\frac{1}{a\sin\alpha}\frac{dV}{d\theta} = -\frac{3Ma}{D^3}(\cos\theta \sin\alpha \sin\beta + \cos\alpha \cos\beta)\sin\theta \sin\beta \ldots (42).$$

We must now express the right-hand side of this equation in terms of nt. We have

$$\begin{aligned} \sin\theta \sin\beta &= \sin\beta \sin(\phi - KpH) \\ &= \sin\phi \sin\beta \cos KpH - \cos\phi \sin\beta \sin KpH. \end{aligned}$$

From the spherical triangle MpP we obtain

$$\cos KpH = -\cos MpP = (\cos\gamma \cos\beta - \sin\delta)/\sin\beta \sin\gamma,$$

whence

$$\begin{aligned} \sin\beta \cos KpH &= \cot\gamma(\cos\delta \sin\gamma \cos nt + \cos\gamma \sin\delta) - \sin\delta \operatorname{cosec}\gamma \\ &= \cos\delta \cos\gamma \cos nt - \sin\delta \sin\gamma \qquad (43), \end{aligned}$$

also $\qquad \sin\beta \sin KpH = \sin\beta \sin MpP = \cos\delta \sin nt \ldots\ldots (44),$

therefore

$$\begin{aligned} \sin\theta \sin\beta = &- \sin\phi \sin\delta \sin\gamma + \sin\phi \cos\delta \cos\gamma \cos nt \\ &- \cos\phi \cos\delta \sin nt \qquad (45). \end{aligned}$$

Again

$$\begin{aligned} \cos\theta \sin\beta &= \cos\phi \sin\beta \cos KpH + \sin\phi \sin\beta \sin KpH \\ &= \cos\phi(\cos\delta \cos\gamma \cos nt - \sin\delta \sin\gamma) + \sin\phi \cos\delta \sin nt, \end{aligned}$$

by (43) and (44); whence

$$\begin{aligned} \cos\theta \sin\alpha \sin\beta + \cos\alpha \cos\beta = &\sin\delta(\cos\alpha \cos\gamma - \sin\alpha \sin\gamma \cos\phi) \\ &+ \cos\delta(\cos\alpha \sin\gamma + \sin\alpha \cos\gamma \cos\phi)\cos nt \\ &+ \sin\phi \sin\alpha \cos\delta \sin nt \qquad (46). \end{aligned}$$

The complete solution of the problem is obtained by expressing the value of the disturbing force in the form $A \sin(pt + q\phi + r)$ by means of (42), (45) and (46), then putting $\phi = x/a \sin \alpha$, and finding the forced oscillation by means of (39) or (40).

450. Let us now suppose that the disturbing body lies on the equator; then $\delta = 0$, and the right-hand sides of (45) and (46) respectively become

$$\cos\gamma \sin\phi \cos nt - \cos\phi \sin nt \ldots\ldots\ldots\ldots\ldots(47),$$

and

$$(\cos\alpha \sin\gamma + \sin\alpha \cos\gamma \cos\phi) \cos nt + \sin\alpha \sin\phi \sin nt \ldots\ldots\ldots(48).$$

The product of these expressions multiplied by $-3Ma/D^3$ will give the disturbing force. This product will be found to consist of three parts, the first of which is independent of t and therefore does not produce any tide but simply alters the mean level of the water. The second part depends upon ϕ and $2nt$; and the third part upon 2ϕ and $2nt$.

Since t enters in the form $2nt$, the tides represented by both terms will be *semi-diurnal;* and we shall first consider the second part which is equal to

$$\tfrac{3}{2}MaD^{-3}(-\cos\alpha \sin\gamma \cos\gamma \sin\phi \cos 2nt + \cos\alpha \sin\gamma \cos\phi \sin 2nt)$$

$$= 3MaD^{-3} \cos\alpha \sin\tfrac{1}{2}\gamma \cos\tfrac{1}{2}\gamma \{\sin^2\tfrac{1}{2}\gamma \sin(2nt+\phi) + \cos^2\tfrac{1}{2}\gamma \sin(2nt-\phi)\}.$$

In order to find the elevation of the water we must put $\phi = x/a \sin\alpha$, and substitute the preceding expression for X in (39); we thus obtain from the second of (40)

$$\eta = \frac{3Ma^2h}{2D^3(gh - 4n^2a^2\sin^2\alpha)} \sin\alpha \cos\alpha \sin\gamma \{\cos^2\tfrac{1}{2}\gamma \cos(2nt - \phi) - \sin^2\tfrac{1}{2}\gamma \cos(2nt + \phi)\},$$

which represents two waves travelling in opposite directions.

If in this expression we put $\tan\psi = \tan\phi \sec\gamma$, it becomes

$$\eta = \frac{3Ma^2h}{2D^3(gh - 4n^2a^2\sin^2\alpha)} \sin\alpha \cos\alpha \sin\gamma (\cos^2\gamma \cos^2\phi + \sin^2\phi)^{\frac{1}{2}} \times \cos(2nt - \psi) \ldots\ldots\ldots\ldots(49).$$

The preceding value of η shows (i) that the oscillation at the place of observation goes through all its phases twice during a complete revolution of the moon, it therefore represents a *semi-diurnal tide;* (ii) that at any particular instant, η goes through all

its values as ϕ changes from 0 to 2π; hence the elevation is different at every point of the canal, and therefore (49) represents a single wave travelling round the canal with an irregular motion twice in a tidal day. Since $\gamma = 0$ when the pole of the canal coincides with the pole of the earth, this tide does not exist when the canal coincides with a parallel of latitude.

451. The oscillation just considered, constitutes what Airy calls the *first semi-diurnal tide;* we must now consider the third part of the disturbing force, which depends upon 2ϕ and $2nt$, and which constitutes the *second semi-diurnal tide.* From (47) and (48) the portion of the disturbing force which depends on these terms will be found to be,

$$\tfrac{3}{2}MaD^{-3}\{\sin\alpha\cos\gamma\cos 2\phi\sin 2nt - \tfrac{1}{2}\sin\alpha(1+\cos^2\gamma)\sin 2\phi\cos 2nt\}$$
$$= \tfrac{3}{2}MaD^{-3}\sin\alpha\{\cos^4\tfrac{1}{2}\gamma\sin(2nt-2\phi) - \sin^4\tfrac{1}{2}\gamma\sin(2nt+2\phi)\},$$

whence the elevation is

$$\eta = \frac{3Ma^2h\sin^2\alpha}{4D^3(gh - n^2a^2\sin^2\alpha)}\{\cos^4\tfrac{1}{2}\gamma\sin(2nt-2\phi) - \sin^4\tfrac{1}{2}\gamma\sin(2nt-2\phi)\}.$$

Putting $\tan\chi = 2\cos\gamma\tan 2\phi/(1+\cos^2\gamma)$, this may be written,

$$\eta = \frac{3Ma^2h\sin^2\alpha}{4D^3(gh - n^2a^2\sin^2\alpha)}\{\tfrac{1}{4}(1+\cos^2\gamma)^2\cos^2 2\phi + \cos^2\gamma\sin^2 2\phi\}^{\frac{1}{2}}$$
$$\times\cos(2nt-\chi)\ldots\ldots\ldots\ldots\ldots(50).$$

The preceding value of η shows (i) that the oscillation at the place of observation goes through all its phases twice during a complete revolution of the moon, hence the tide is semi-diurnal; (ii) that the height of the tide at a point $\pi+\phi$ is the same as that at a point ϕ: hence there is a double wave on the canal which travels round the canal with an irregular motion *once* in a tidal day.

452. The waves which we have investigated in §§ 450—1 compound into a single wave at the place of observation; for they are each represented by terms of the form $A\cos(2nt-\psi)$ and $B\cos(2nt-\chi)$ which may evidently be compounded into a single term of the form $C\cos(2nt-\Omega)$. The quantities C and Ω depend upon the dimensions and position of the canal; hence the magnitude of the tide and other special circumstances connected with it cannot be investigated without a knowledge of their values.

453. Let us now suppose that the canal is a great circle, whilst projection of the path of the disturbing body is any small circle. In this case $\alpha = \frac{1}{2}\pi$, and the value of $\sin\theta\sin\beta$ is given by (45), and the right-hand side of (46) becomes

$$-\sin\gamma\sin\delta\cos\phi + \cos\gamma\cos\delta\cos\phi\cos nt + \cos\delta\sin\phi\sin nt \ldots (51).$$

If the right-hand sides of (45) and (51) be multiplied together, and the result multiplied by $-3Ma/D^3$, we shall find that the disturbing force consists of three parts. The first part is independent of nt, and shows that the mean elevation is modified by the action of the disturbing body. The second part depends upon 2ϕ and nt, and the third on 2ϕ and $2nt$.

The tides produced by these terms can be investigated in precisely the same manner as in §§ 450—1, and it will be found that the height of the tide produced by the terms depending on 2ϕ and nt is

$$\eta = -\frac{3Ma^2h}{D^3(4gh - n^2a^2)}\sin 2\delta\sin\gamma(\cos^2\gamma\cos^2 2\phi + \sin^2 2\phi)^{\frac{1}{2}}$$
$$\times\cos(nt - \psi) \ldots\ldots\ldots\ldots\ldots (52),$$

where $\tan\psi = \sec\gamma\tan 2\phi$. This tide is therefore a *diurnal tide;* also since the value of η at the point $\pi + \phi$ is the same as at the point ϕ, there are two waves in the canal, each of which travels round the canal with an irregular motion once in two days. Since the elevation depends upon $\sin 2\delta$, it changes sign when the luminary crosses the equator, and vanishes when the luminary is on the equator. If therefore the path of the disturbing body coincides with the equator, this tide vanishes.

If the canal coincides with a meridian, $\gamma = \frac{1}{2}\pi$, and (52) becomes

$$\eta = -\frac{3Ma^2h}{D^3(4gh - n^2a^2)}\sin 2\delta\sin 2\phi\sin nt \ldots\ldots\ldots (53),$$

hence the wave is a stationary wave, whose period is *diurnal.* The elevation vanishes at the poles where $\phi = 0$ or π, and at the equator where $\phi = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$; also an elevation in north latitude occurs at the same time as a depression in south latitude, and the tide will be highest (or lowest) in lat. 45°. The sign of η will depend on that of $4gh - n^2a^2$, which depends on the depth of the canal.

If the canal is equatorial $\gamma = 0$, and therefore the tide vanishes.

454. The portion of the disturbing force depending on 2ϕ and $2nt$ can be shown as in § 451 to produce an elevation

$$\eta = \frac{3Ma^2h\cos^2\delta}{4D^3(gh-n^2a^2)}\{\tfrac{1}{4}(1+\cos^2\gamma)^2\cos^2 2\phi + \cos^2\gamma\sin^2 2\phi\}^{\frac{1}{2}}$$
$$\times \cos(2nt-\chi) \quad\ldots\ldots\ldots\ldots\ldots(54),$$

where $\tan\chi = 2\cos\gamma\tan 2\phi/(1+\cos^2\gamma)$. Hence the portion of the tide which depends on these terms is *semi-diurnal*, and consists of two waves on the canal travelling round it with an irregular motion once a day.

Since the declination of the sun or moon is never equal to 90°, this tide can never vanish for any position of the disturbing body.

If the canal is equatorial, $\gamma = 0$, and (54) becomes

$$\eta = \frac{3Ma^2h\cos^2\delta}{4D^3(gh-n^2a^2)}\cos(2nt-2\phi) \quad\ldots\ldots\ldots\ldots(55).$$

Hence the tide will be direct or inverted according as $h >$ or $< n^2a^2/g$.

If the canal passes through the pole $\gamma = \frac{1}{2}\pi$, whence

$$\eta = \frac{3Ma^2h\cos^2\delta}{8D^3(gh-n^2a^2)}\cos 2\phi\cos 2nt \quad\ldots\ldots\ldots\ldots(56),$$

which represents a stationary wave.

455. If the period of apparent revolution of the disturbing body round the pole were exactly equal to the period of rotation of the earth, which is very nearly true in the case of the sun, though less so in the case of the moon, n^2a/g would be equal to $\frac{1}{289}$; and therefore the denominator of (55) would be negative if $h < a/289$, or < 14 miles about. Now the depths of the oceans which cover the earth are less than 14 miles, it therefore follows that when the luminary is on the meridian of the place of observation, or $nt = \phi$, the tide considered in (55) will be inverted.

If in (50) the canal coincides with a parallel of latitude, $\gamma = 0$, and $\chi = 2\phi$, hence the tide will be inverted unless $h > 14\sin^2\alpha$ where h is the depth of the canal in miles. At the equator $\alpha = \frac{1}{2}\pi$ and at the poles $\alpha = 0$, it therefore follows that whatever the depth of the ocean may be there must be a certain latitude for which this tide vanishes, which is equal to $\cos^{-1}(h/14)^{\frac{1}{2}}$, and therefore in higher latitudes the tide will be direct, whilst in lower latitudes the tide will be inverted.

The first portion of the disturbing force in the preceding sections, which does not contain nt, is not absolutely constant, since it depends upon the motion of the moon about the earth, or of the earth about the sun, according as the disturbing body is the moon or the sun. Hence these terms will give rise to tides of long period; we shall not however investigate them but refer the reader to Airy's treatise[1] where they are discussed.

Tides in Estuaries.

456. We have shown in § 403, that the equation of motion for long waves is

$$\frac{d^2\xi}{dt^2} = v^2 \frac{d^2\xi}{dx^2}\left(1 + \frac{d\xi}{dx}\right)^{-3} \quad \text{..................(57)},$$

and that the elevation η is

$$\eta = -h\frac{d\xi}{dx}\left(1 + \frac{d\xi}{dx}\right)^{-1} \quad \text{..................(58)},$$

where h is the depth of the water and $v^2 = gh$.

Let us now suppose that a gulf or tidal river communicates with the sea. Owing to the tides in the sea, there will be a tide in the river up to a certain point; also if the length of the river be short in comparison with the radius of the earth, the tides produced by the direct action of the sun and moon will be small in comparison with the tides produced by the rise and fall of the ocean with which the river communicates. The elevation of the water at the mouth of the river may be represented by a term of the form $\eta = H \sin nt$, and the problem consists in finding the forced oscillations of the river due to this term.

Since η and therefore $d\xi/dx$ are small, (57) may be written

$$\frac{d^2\xi}{dt^2} = v^2 \frac{d^2\xi}{dx^2}\left(1 - 3\frac{d\xi}{dx}\right) \quad \text{..................(59)}.$$

For a first approximation omit the last term on the right-hand side of (59), and we obtain on integration,

$$\xi = a\cos m(vt - x), \quad \eta = -mah\sin m(vt - x),$$

where $m = n/v$, $H = -mah$.

[1] "Tides and Waves," Sec. VI. §§ 446—449.

The preceding value of η gives the height of the tide at any point up the river to a first approximation. In order to obtain a second approximation, substitute the preceding value of ξ in the last term on the right-hand side of (59) and we obtain putting $u = vt - x$,

$$\frac{d^2\xi}{dt^2} = v^2 \frac{d^2\xi}{dx^2} + \tfrac{3}{2}v^2a^2m^3 \sin 2mu.$$

In order to solve this equation, assume

$$\xi = a \cos mu + Ax \cos 2mu + B \sin 2mu$$

and we find $A = -\tfrac{3}{8}a^2m^2$; and from (58) we obtain

$$\eta/h = \tfrac{1}{2}m^2a^2 - ma \sin mu + \tfrac{3}{4}m^3a^2x \sin 2mu$$
$$+ (2mB - \tfrac{1}{8}m^2a^2) \cos 2mu \ldots (60).$$

Since $\eta = H \sin nt$ when $x = 0$, we must have

$$B = -\tfrac{3}{16}ma^2,$$

and therefore

$$\eta/h = -ma \sin mu + \tfrac{3}{4}m^3a^2x \sin 2mu + \tfrac{1}{2}m^2a^2(1 - \cos 2mu) \ldots (61).$$

457. In the preceding investigation we have implicitly assumed that the terms involving $2mu$ are small in comparison with those involving mu. Now the coefficient of $\cos 2mu$ is Ax, and this will not be small if x is large; but in order to evade this difficulty we may take the canal of finite length, and suppose that the other extremity is connected with a large lake at which an appropriate forced oscillation is maintained.

The first term of (61) is called the *fundamental* or *oceanic tide;* and the second is called the first *over-tide.* The velocities of propagation of the two tides are the same, but the *frequency* or *speed* of the latter is double that of the former. It also appears that the times of high and low tide are the same throughout the estuary.

As a matter of fact the time of high tide in a tidal river differs at different places. For example, if it is high tide at Margate at noon, high tide at Gravesend occurs at a quarter past two, and at London Bridge a few minutes before three; hence the preceding results can scarcely be considered an approximate representation of the facts. Of course the tides in an estuary depend largely upon its form, the presence of shoals and other causes; also the effect of the viscosity of the water, and the friction against the bed of the

estuary due to the inequalities of the latter, must materially influence the motion. The solution of the problem when friction is taken into account has been given by Airy, upon the supposition that the effect of friction may be represented by a term proportional to the velocity, and may therefore be obtained by adding the term $\mu d\xi/dt$ to the left-hand side of (57) and proceeding as before; and the form of his solution shows that the tide gradually travels up the river, which is in better agreement with the facts. For further information on this point, we must refer the reader to Airy's *Tides and Waves*, and to Prof. Darwin's article on *Tides* in the *Encyclopædia Britannica.*

CHAPTER XX.

ON THE GENERAL EQUATIONS OF MOTION OF A VISCOUS FLUID.

458. We have defined a *perfect* fluid to be one which is incapable of sustaining any tangential stress, and have shown as a necessary consequence of this definition, that whether such a fluid is at rest or in motion the pressure at every point is the same in all directions, and acts in a direction perpendicular to every plane through that point. We have also pointed out that this condition is not fulfilled in the case of any fluid which exists in nature, since every fluid with which we are acquainted is capable of sustaining tangential stresses, and consequently the pressure at a point is not perpendicular to every plane drawn through that point, neither is it the same in all directions.

It further appears from experiment that whenever a fluid is set in motion and then left to itself, the motion gradually subsides and ultimately dies away, and an apparent loss of energy takes place. This apparent loss of energy is due to the internal friction of the fluid, which causes the kinetic energy of the motion to be converted into heat.

Various theories[1] have been constructed to explain the nature

[1] Navier, *Mém. de l'Acad. des Sciences*, vol. VI. p. 389.

Poisson, *Journal de l'École Polytechnique*, vol. XIII. p. 139.

Barré de Saint-Venant, *Comptes Rendus*, vol. XVII. p. 1240.

A description of these three papers is given by Stokes, *Brit. Assoc. Rep. Hydrodynamics*, 1846. See also,

Meyer, Ueber die Reibung der Flüssigkeiten, *Borch*, vol. LIX. p. 229; and vols. LXXVIII. p. 130, and LXXX. p. 315.

Stefan, Ueber die Bewegung flüssiger Körper, *Sitz. Akad. Wiss. Wien*, vol. XLVI. p. 8.

Maxwell, "On the dynamical theory of gases," *Phil. Trans.* 1867, p. 81; and *Phil. Mag.* Jan. and July, 1860.

Levy, *Comptes Rendus*, vol. LXVIII. p. 582.

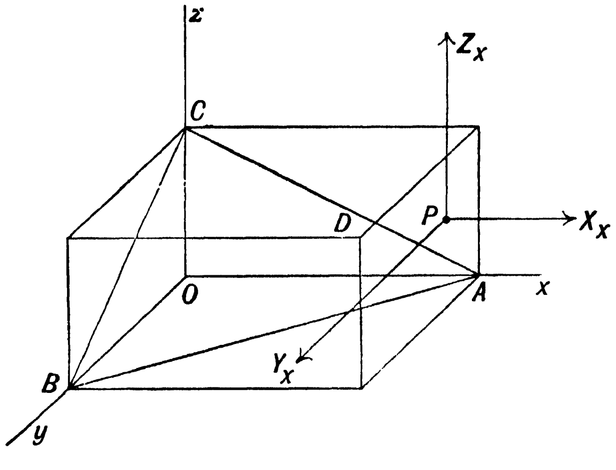
Kleitz, *Ibid.* vol. LXXIV. p. 426.

Butcher, "On Viscous Fluids in Motion," *Proc. Lond. Math. Soc.* vol. VIII. p. 103.

A description of these latter papers is given by Hicks, *Brit. Assoc. Rep. Hydrodynamics*, 1881—2.

and effects of fluid friction, some of which depend upon speculations concerning the molecular constitution of matter; none of them can be regarded as altogether satisfactory, although they furnish results which experiment proves to be true when the motion of the fluid is slow. The theory which will be explained in the present chapter is due to Prof. Stokes[1], and depends partly on the theory of the internal stresses which are experienced by media which are capable of resisting compression and distortion, and partly upon three assumptions.

459. The general theory of the internal stresses experienced by a medium which is capable of resisting compression and distortion, is given in treatises on Elasticity; but for the sake of completeness, it will be desirable to give an outline of this theory, so far as is necessary for our present purpose. We shall therefore commence by examining the stresses which act upon an element of such a medium.



Let the figure represent a small parallelopiped of the medium. The stresses which act on the face AD are,

(i) A normal stress or traction X_x parallel to Ox;

(ii) A tangential stress or shear Y_x parallel to Oy;

(iii) A tangential stress or shear Z_x parallel to Oz.

Similarly the remaining stresses which act on the faces BD and CD are Y_y, Z_y, X_y and Z_z, X_z, Y_z.

These are the stresses exerted on the faces AD, BD, CD of the element by the surrounding medium; the stresses exerted by the medium on the three opposite faces will be in the opposite directions.

[1] *Trans. Camb. Phil. Soc.* vol. VIII. p. 287.

460. Let F, G, H be the components parallel to the axes, of the stresses upon the plane ABC, whose area is Δ and whose direction cosines are l, m, n. The conditions of equilibrium of the tetrahedron $OABC$ require that

$$F\Delta = (lX_x + mX_y + nX_z)\,\Delta.$$

We thus obtain

$$\left.\begin{aligned} F &= lX_x + mX_y + nX_z \\ G &= lY_x + mY_y + nY_z \\ H &= lZ_x + mZ_y + nZ_z \end{aligned}\right\} \quad\text{.....................(1).}$$

461. The preceding results are true of any medium which is capable of resisting compression and distortion. We shall now suppose that the medium is a viscous fluid, and shall proceed to find the equations of motion.

Let X, Y, Z be the components per unit of mass, of the impressed forces which act on the fluid; ρ its density, and q its resultant velocity. Describe any imaginary fixed surface S in the fluid, and let ϵ be the angle which the direction of q makes with the normal to S drawn outwards.

The rate of increase of the component of momentum parallel to x of the fluid contained within S, is equal to the rate at which momentum parallel to x flows into S across the boundary of S, together with the rate at which momentum parallel to x is generated by the component of the impressed force parallel to x, and by the component parallel to x of the stresses exerted by the surrounding fluid upon the boundary of S[1].

[1] The principle upon which this method depends was erroneously stated in Vol. I. § 21. The correct principle for a frictionless fluid is, as stated above, with *pressure* substituted for *stresses;* lines 13 and 14 of page 21 should therefore be,

The rate at which momentum parallel to x flows into S, is

$$-\iint \rho q u \cos \epsilon\, dS = -\iint \rho u\,(lu + mv + nw)\, dS\,;$$

using this in § 21 together with the principle stated above and taking account of the equation of continuity, we shall obtain the equations of motion of a frictionless fluid in their ordinary form.

A similar modification is required in § 35. In this case the principle is;

The rate of increase of the kinetic energy of the fluid contained within S, is equal to the rate at which kinetic energy flows into S across its boundary, together with the rate at which work is done upon the fluid contained within S by the impressed forces, and by the pressure of the surrounding fluid upon the boundary of S. Lines 6 and 5 from the bottom of page 31 should therefore be,

The rate at which kinetic energy flows into S,

$$= -\iint \rho T\,(lu + mv + nw)\, dS.$$

The rate of increase of the component momentum parallel to x of the fluid contained within S, is

$$\iiint \frac{d}{dt}(\rho u)\, dx\, dy\, dz.$$

The rate at which momentum parallel to x flows into S, is

$$-\iint \rho q u \cos \epsilon\, dS = -\iint \rho u (lu + mv + nw)\, dS$$
$$= -\iiint \left\{\frac{d(\rho u^2)}{dx} + \frac{d(\rho uv)}{dy} + \frac{d(\rho uw)}{dz}\right\} dx\, dy\, dz,$$

by § 7.

The rate at which momentum parallel to x is generated by the impressed forces, is

$$\iiint \rho X\, dx\, dy\, dz.$$

The rate at which momentum parallel to x is generated by the stresses exerted by the surrounding fluid upon the boundary of S, is

$$\iint (lX_x + mX_y + nX_z)\, dS = \iiint \left(\frac{dX_x}{dx} + \frac{dX_y}{dy} + \frac{dX_z}{dz}\right) dx\, dy\, dz,$$

by § 7. Whence

$$\iiint \left\{\frac{d}{dt}(\rho u) + \frac{d(\rho u^2)}{dx} + \frac{d(\rho uv)}{dy} + \frac{d(\rho uw)}{dz}\right\} dx\, dy\, dz$$
$$= \iiint \left(\rho X + \frac{dX_x}{dx} + \frac{dX_y}{dy} + \frac{dX_z}{dz}\right) dx\, dy\, dz,$$

whence reducing S to a point, we obtain

$$\frac{d(\rho u)}{dt} + \frac{d(\rho u^2)}{dx} + \frac{d(\rho uv)}{dy} + \frac{d(\rho uw)}{dz} = \rho X + \frac{dX_x}{dx} + \frac{dX_y}{dy} + \frac{dX_z}{dz}.$$

Taking account of the equation of continuity, and of the other two equations which can be obtained by considering the rates of increase of the component momenta parallel to y and z, we obtain the equations of motion in the form

$$\left.\begin{aligned}
\rho \frac{\partial u}{\partial t} &= \rho X + \frac{dX_x}{dx} + \frac{dX_y}{dy} + \frac{dX_z}{dz} \\
\rho \frac{\partial v}{dt} &= \rho Y + \frac{dY_x}{dx} + \frac{dY_y}{dy} + \frac{dY_z}{dz} \\
\rho \frac{\partial w}{\partial t} &= \rho Z + \frac{dZ_x}{dx} + \frac{dZ_y}{dy} + \frac{dZ_z}{dz}
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2).$$

462. In addition to the equations expressing the fact that the rate of increase of the *linear* momentum within a closed space is due to the causes above mentioned, we must also express in an analytical form the fact that the rate of increase of the *moment of momentum* of the fluid within S about any axis, is equal to the rate at which moment of momentum about this axis is brought in by the fluid crossing the boundary, together with the rate at which moment of momentum is generated by the forces which act upon this portion of the fluid. This will enable us to show that,

$$X_y = Y_x,\ Y_z = Z_y,\ Z_x = X_z \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The rate of increase of the moment of momentum about x, of the fluid contained within S, is

$$\iiint \left(y\frac{d(\rho w)}{dt} - z\frac{d(\rho v)}{dt}\right) dx\,dy\,dz.$$

The rate at which moment of momentum flows into S, is

$$-\iint \rho\,(lu + mv + nw)(yw - zv)\,dS.$$

The rate at which moment of momentum is generated by the impressed forces is

$$\iiint \rho\,(yZ - zY)\,dx\,dy\,dz\,;$$

and the rate at which it is generated by the surface stresses is

$$\iint \{y(lZ_x + mZ_y + nZ_z) - z(lY_x + mY_y + nY_z)\}\,dS.$$

Transforming the surface integrals into volume integrals by § 7, and making use of the equation of continuity, we shall obtain

$$\iiint y\left(\rho\frac{\partial w}{\partial t} - \rho Z - \frac{dZ_x}{dx} - \frac{dZ_y}{dy} - \frac{dZ_z}{dz}\right) dx\,dy\,dz$$

$$-\iiint z\left(\rho\frac{\partial v}{\partial t} - \rho Y - \frac{dY_x}{dx} - \frac{dY_y}{dy} - \frac{dY_z}{dz}\right) dx\,dy\,dz + \iiint (Z_y - Y_z)\,dx\,dy\,dz = 0.$$

From (2) it follows that the first two integrals vanish, whence $Z_y = Y_z$, and similarly $Z_x = X_z$ and $Y_x = X_y$.

463. From the preceding investigation it appears that the components of stress are completely specified by the six quantities X_x, Y_y, Z_z, Y_z, Z_x, X_y, which we shall in future denote by the letters P, Q, R, S, T, U. Equations (1) and (2) may now be written

$$\left.\begin{aligned} F &= Pl + Um + Tn \\ G &= Ul + Qm + Sn \\ H &= Tl + Sm + Rn \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and
$$\left.\begin{aligned}\rho\frac{\partial u}{\partial t}&=\rho X+\frac{dP}{dx}+\frac{dU}{dy}+\frac{dT}{dz}\\ \rho\frac{\partial v}{\partial t}&=\rho Y+\frac{dU}{dx}+\frac{dQ}{dy}+\frac{dS}{dz}\\ \rho\frac{\partial w}{\partial t}&=\rho Z+\frac{dT}{dx}+\frac{dS}{dy}+\frac{dR}{dz}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(5).$$

464. It appears from (4) that if we construct the quadric

$$Px^2+Qy^2+Rz^2+2Syz+2Tzx+2Uxy=1\ldots\ldots(6),$$

then F, G, H will be proportional to the direction cosines λ, μ, ν of the normal to this quadric at the point rl, rm, rn, hence

$$F=\lambda/rp,\quad G=\mu/rp,\quad H=\nu/rp.$$

and
$$F^2+G^2+H^2=(rp)^{-2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

where p is the perpendicular from the centre of the quadric on to the tangent plane at rl, rm, rn.

Hence the magnitude and direction of the stress across any plane may be found by the following construction.

From the centre of the stress quadric (6), draw a line perpendicular to the plane and meeting the quadric at P, draw the tangent plane at P; *then the required stress will be in the direction of the perpendicular on to the tangent plane at P, and will be equal to the reciprocal of the product of this perpendicular and the radius vector to P.*

If the stress quadric be referred to its principal axes, its equation will be of the form

$$P'x^2+Q'y^2+R'z^2=1,$$

where P', Q', R' are the normal tractions perpendicular to the three co-ordinate planes. It thus appears that the tangential stresses across these planes are zero; hence *there are always three planes mutually at right angles to one another, such that the stresses across these three planes are altogether perpendicular to them.*

465. If F', G', H' are the stresses perpendicular to any other three planes mutually at right angles to one another, whose direction cosines referred to the principal axes of the stress quadric are (l, m, n), (λ, μ, ν), (L, M, N), we obtain from (4)

$$F' = P'l^2 + Q'm^2 + R'n^2$$
$$G' = P'\lambda^2 + Q'\mu^2 + R'\nu^2$$
$$H' = P'L^2 + Q'M^2 + R'N^2,$$

whence $$F' + G' + H' = P' + Q' + R' \quad \text{..................(8)}.$$

Hence the sum of the three normal stresses across any three planes mutually at right angles to each other is constant.

466. Equations (5) have been established by perfectly rigorous dynamical methods, but before any use can be made of them, it is necessary to connect the six components of stress with the velocities; and in order to do this the first assumption has to be made.

Let u, v, w be the velocities of the centre of inertia G of any small element of the fluid; let x, y, z be its co-ordinates and $x+x'$, $y+y'$, $z+z'$ those of a point P near G. The component velocities of P are

$$\left.\begin{aligned} u' &= u + x'\frac{du}{dx} + y'\frac{du}{dy} + z'\frac{du}{dz} \\ v' &= v + x'\frac{dv}{dx} + y'\frac{dv}{dy} + z'\frac{dv}{dz} \\ w' &= w + x'\frac{dw}{dx} + y'\frac{dw}{dy} + z'\frac{dw}{dz} \end{aligned}\right\} \quad \text{..............(9)}.$$

If we put

$$\left.\begin{aligned} & e = \frac{du}{dx},\ f = \frac{dv}{dy},\ g = \frac{dw}{dz}, \\ & a = \tfrac{1}{2}\left(\frac{dw}{dy} + \frac{dv}{dz}\right),\ b = \tfrac{1}{2}\left(\frac{du}{dz} + \frac{dw}{dx}\right),\ c = \tfrac{1}{2}\left(\frac{dv}{dx} + \frac{du}{dy}\right) \end{aligned}\right\} \quad \text{...(10)},$$

equations (9) may be written

$$\left.\begin{aligned} u' &= u + ex' + cy' + bz' + \eta z' - \zeta y' \\ v' &= v + cx' + fy' + az' + \zeta x' - \xi z' \\ w' &= w + bx' + ay' + gz' + \xi y' - \eta x' \end{aligned}\right\} \quad \text{..........(11)},$$

where ξ, η, ζ, as usual, denote the components of molecular rotation.

The first term of each equation represents a motion of translation of the whole element of fluid.

The next three terms represent a motion, such that every point on the surface of the quadric

$$ex^2 + fy^2 + gz^2 + 2ayz + 2bzx + 2cxy = 1,$$

is moving in the direction of the normal at that point. If this quadric is referred to its principal axes, its equation will be of the form

$$e'x^2 + f'y^2 + g'z^2 = 1,$$

and the corresponding portions of the component velocities will be

$$u' = e'x,\ v' = f'y,\ w = g'z \qquad\qquad (12).$$

Equation (12) shows that every line of the element parallel to the axes is being elongated (or contracted) at the rates e', f', g' respectively. This kind of motion is called a pure strain or distortion; and the six quantities a, b, c, e, f, g, are the six components of the *rate of strain.*

The last two terms of (11) represent a motion of rotation of the element, whose component angular velocities are ξ, η, ζ.

Hence the motion of every small element of fluid consists;

(i) of a motion of translation of the whole element;

(ii) a motion of distortion;

(iii) a motion of rotation about an instantaneous axis.

Now the internal friction of a fluid in motion is caused by the different elements of the fluid rubbing against one another. In the case of a perfectly rigid body no such rubbing takes place, and there is no internal friction; and since the parts (i) and (iii) of the motion of the element are such as belong to a rigid body, it is inferred that these parts of the motion cannot give rise to internal friction, which is therefore due to the motion of distortion. Hence the first assumption is that

The six stresses due to viscosity depend solely on the motion of distortion, and are therefore functions of the six components of the rate of strain.

If the velocity of the fluid is small, e, f, g, a, b, c, will all be small quantities, and therefore if we expand the stresses in terms of the rates of strain and neglect squares and higher powers of small quantities, the stresses will be linear functions of the rates of strain.

The second assumption will therefore be that; *The six stresses due to viscosity are linear functions of the rates of strain, the coefficients of which are all constant quantities, which depend on the viscosity of the fluid.*

Since this assumption depends upon the supposition that the velocity is small, it is not of an altogether satisfactory character, when the velocity is not small.

Since the tangential stresses S, T, U are zero when the fluid is frictionless, they must depend entirely on the viscosity, and therefore cannot contain any terms independent of the rates of strain; but the normal stresses P, Q, R do not vanish when the fluid is frictionless, but are each equal to $-p$, where p is the pressure. These stresses are therefore composed of two parts, one of which is a linear function of the rates of strain, and the other of which is equal to $-p$, where p is a function of x, y, z and t, which is equal to the pressure when the fluid is frictionless; and the third assumption is that;

When a gas is expanding equally in all directions, the stresses P, Q, R are the same as if the fluid were frictionless, and are therefore each equal to $-p$.

We shall therefore assume that P, Q and R are each of the form $-p+P'$, $-p+Q'$, $-p+R'$, where P', Q', R' are linear functions of the rates of strain.

467. Let W be the rate at which work is done per unit of volume by the strains, then δW is the rate at which work must be done in order to change the rates of strain from a, b &c. to $a+\delta a$ &c.; hence from (10)

$$\delta W = P'\delta e + Q'\delta f + R'\delta g + 2(S\delta a + T\delta b + U\delta c)\ldots(13).$$

Since W must be a definite function of the rates of strain, the right-hand side of (13) must be a perfect differential, hence W must be a homogeneous quadratic function of the rates of strain; and therefore in its most general form will contain twenty-one coefficients. But since the fluid is isotropic, W will remain unchanged when $-z$ and $-w$ are written for z and w. This alteration changes a and b into $-a$ and $-b$. Similar observations apply to the planes (xz) and (yz), whence W must be of the form

$$W = \tfrac{1}{2}(Ee^2 + Ff^2 + Gg^2 + Aa^2 + Bb^2 + Cc^2 \\ + 2Lfg + 2Mge + 2Nef)\ldots\ldots\ldots(14),$$

hence
$$\left.\begin{aligned} P' &= \frac{dW}{de} = Ee + Nf + Mg \\ Q' &= \frac{dW}{df} = Ne + Ff + Lg \\ R' &= \frac{dW}{dg} = Me + Lf + Gg \\ 2S &= \frac{dW}{da} = Aa \\ 2T &= \frac{dW}{db} = Bb \\ 2U &= \frac{dW}{dc} = Cc \end{aligned}\right\} \ldots\ldots\ldots\ldots(15).$$

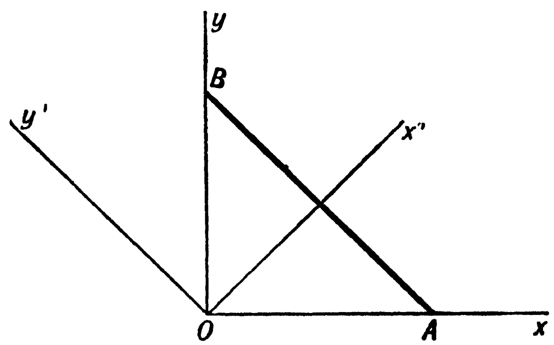
Since the fluid is isotropic we at once obtain $A = B = C$. Also if the stresses are such as to produce a strain e, then $Q' = R'$, therefore $M = N$; similarly $N = L$, whence $L = M = N$. Also if the stresses consist of a single traction P', we must have $f = g$, therefore $F = G$; similarly $G = E$, whence $E = F = G$. Changing the constants and remembering that $P = -p + P'$, we obtain

$$\left.\begin{aligned} P &= -p + \lambda\theta + 2\mu e \\ Q &= -p + \lambda\theta + 2\mu f \\ R &= -p + \lambda\theta + 2\mu g \\ S &= 2ka, \quad T = 2kb, \quad U = 2kc \end{aligned}\right\} \ldots\ldots\ldots\ldots(16),$$

where $\theta = e + f + g$.

In order to obtain the relation between k and μ, let us consider the motion of a fluid in two dimensions.

Let AB be a line meeting the axis of x at an angle $\frac{1}{4}\pi$.



Let u', v' be the velocities of the fluid perpendicular and parallel to AB. Then from (5) and (16)

$$U' = 2^{-\frac{1}{2}}(G - F) = \tfrac{1}{2}(Q - P),$$

whence $$k\left(\frac{du'}{dy'} + \frac{dv'}{dx'}\right) = \mu(f - e) = \mu\left(\frac{dv}{dy} - \frac{du}{dx}\right).$$

But

$$x'\surd 2 = x + y, \quad y'\surd 2 = y - x, \quad u'\surd 2 = u + v, \quad v'\surd 2 = v - u,$$

whence $$\frac{du'}{dy'} + \frac{dv'}{dx'} = \frac{dv}{dy} - \frac{du}{dx},$$

and therefore $$k = \mu \quad \text{.............................(17)}.$$

In the case of a liquid $\theta = 0$, and therefore the terms involving λ disappear, and the third assumption is not required: hence all the components of stress are given by (16) and (17) in terms of the rates of strain, and therefore of the velocities and a quantity μ which depends upon the viscosity of the particular liquid under consideration.

But if the fluid be a gas θ does not vanish, and we therefore require a relation between λ and μ. This is furnished by the third assumption, which asserts that when $e = f = g$, $P = Q = R = -p$; which requires that,

$$3\lambda + 2\mu = 0 \quad \text{........................ (18)},$$

which gives the relation between λ and μ in the case of a gas.

We therefore finally obtain

$$\left.\begin{aligned} P &= -p - \tfrac{2}{3}\mu\theta + 2\mu\frac{du}{dx} \\ Q &= -p - \tfrac{2}{3}\mu\theta + 2\mu\frac{dv}{dy} \\ R &= -p - \tfrac{2}{3}\mu\theta + 2\mu\frac{dw}{dz} \\ S = \mu\left(\frac{dw}{dy} + \frac{dv}{dz}\right), \quad T &= \mu\left(\frac{du}{dz} + \frac{dw}{dx}\right), \quad U = \mu\left(\frac{dv}{dx} + \frac{du}{dy}\right) \end{aligned}\right\} \text{...(19)},$$

and the value of W becomes

$$W = -\tfrac{1}{3}\mu\theta^2 + \mu\{e^2 + f^2 + g^2 + 2(a^2 + b^2 + c^2)\} \text{......(20)}.$$

468. We can now obtain the equations of motion of a viscous fluid in the required form, for substituting the values of P, Q, R, S, T, U from (19) in (5) and putting $\mu/\rho = \nu$, the result is

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= X - \frac{1}{\rho}\frac{dp}{dx} + \tfrac{1}{3}\nu\frac{d\theta}{dx} + \nu\nabla^2 u\\ \frac{\partial v}{\partial t} &= Y - \frac{1}{\rho}\frac{dp}{dy} + \tfrac{1}{3}\nu\frac{d\theta}{dy} + \nu\nabla^2 v\\ \frac{\partial w}{\partial t} &= Z - \frac{1}{\rho}\frac{dp}{dz} + \tfrac{1}{3}\nu\frac{d\theta}{dz} + \nu\nabla^2 w\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (21).$$

When the fluid is incompressible $\theta = 0$, and (21) becomes

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= X - \frac{1}{\rho}\frac{dp}{dx} + \nu\nabla^2 u\\ \frac{\partial v}{\partial t} &= Y - \frac{1}{\rho}\frac{dp}{dy} + \nu\nabla^2 v\\ \frac{\partial w}{\partial t} &= Z - \frac{1}{\rho}\frac{dp}{dz} + \nu\nabla^2 w\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (22),$$

and the values of the stresses are obtained from (19) by putting

$$\theta = 0.$$

469. The constant μ is called the *coefficient of viscosity* of the fluid; it is independent of the pressure and its value is different for different fluids, and can only be found by experiment.

The quantity $\nu = \mu/\rho$ is called the *kinematic coefficient of viscosity.*

470. We shall hereafter require the equations of motion of a viscous liquid referred to cylindrical and polar coordinates.

When cylindrical coordinates are employed, let u', v' be the velocities of the liquid in the directions of x and y; u, v the velocities in the directions of ϖ and θ; then if V be the potential of the impressed forces, and if $Q = -V - p/\rho$, we have

$$u' = u\cos\theta - v\sin\theta,\quad v' = u\sin\theta + v\cos\theta.$$

Also if f_x denote the acceleration parallel to x,

$$f_\varpi = f_x\cos\theta + f_y\sin\theta = \frac{dQ}{d\varpi} + \nu\cos\theta\nabla^2 u' + \nu\sin\theta\nabla^2 v',$$

$$f_\theta = f_y\cos\theta - f_x\sin\theta = \frac{1}{\varpi}\frac{dQ}{d\theta} + \nu\cos\theta\nabla^2 v' - \nu\sin\theta\nabla^2 u'.$$

Now

$$\nabla^2 u' = \cos\theta\nabla^2 u - \sin\theta\nabla^2 v - \frac{2}{\varpi^2}\sin\theta\frac{du}{d\theta}$$
$$- \frac{u}{\varpi^2}\cos\theta - \frac{2}{\varpi^2}\cos\theta\frac{dv}{d\theta} + \frac{v}{\varpi^2}\sin\theta$$

$$\nabla^2 v' = \sin\theta\nabla^2 u + \cos\theta\nabla^2 v + \frac{2}{\varpi^2}\cos\theta\frac{du}{d\theta}$$
$$- \frac{u}{\varpi^2}\sin\theta - \frac{2}{\varpi^2}\sin\theta\frac{dv}{d\theta} - \frac{v}{\varpi^2}\cos\theta.$$

Therefore $\cos\theta\nabla^2 u' + \sin\theta\nabla^2 v' = \nabla^2 u - \frac{u}{\varpi^2} - \frac{2}{\varpi^2}\frac{dv}{d\theta}$,

$$\cos\theta\nabla^2 v' - \sin\theta\nabla^2 u' = \nabla^2 v - \frac{v}{\varpi^2} + \frac{2}{\varpi^2}\frac{du}{d\theta}.$$

Substituting the values of f_ϖ and f_θ from (7) of § 6, we obtain

$$\left.\begin{aligned}
\frac{\partial u}{\partial t} - \frac{v^2}{\varpi} &= \frac{dQ}{d\varpi} + \nu\left(\nabla^2 u - \frac{u}{\varpi^2} - \frac{2}{\varpi^2}\frac{dv}{d\theta}\right)\\
\frac{\partial v}{\partial t} + \frac{uv}{\varpi} &= \frac{1}{\varpi}\frac{dQ}{d\theta} + \nu\left(\nabla^2 v - \frac{v}{\varpi^2} + \frac{2}{\varpi^2}\frac{du}{d\theta}\right)\\
\frac{\partial w}{\partial t} \quad &= \frac{dQ}{dz} + \nu\nabla^2 w
\end{aligned}\right\} \quad \dots\dots\dots (23),$$

where
$$\frac{\partial}{\partial t} = \frac{d}{dt} + u\frac{d}{d\varpi} + \frac{v}{\varpi}\frac{d}{d\theta} + w\frac{d}{dz}.$$

In order to obtain the equations referred to polar coordinates r, θ, ϕ, we must recollect that the θ in cylindrical coordinates is the ϕ in polar coordinates. Let U, V, W be the velocities in the directions r, θ, ϕ; then

$$u = U\sin\theta + V\cos\theta,\ w = U\cos\theta - V\sin\theta,\ v = W.$$

From (23) we obtain

$$f_r = f_z\cos\theta + f_\varpi\sin\theta = \frac{dQ}{dr} + \nu\cos\theta\,\nabla^2 w + \nu\sin\theta\left(\nabla^2 u - \frac{u}{\varpi^2} - \frac{2}{\varpi^2}\frac{dW}{d\phi}\right)$$

$$f_\theta = f_\varpi\cos\theta - f_z\sin\theta = \frac{1}{r}\frac{dQ}{d\theta} + \nu\cos\theta\left(\nabla^2 u - \frac{u}{\varpi^2} - \frac{2}{\varpi^2}\frac{dW}{d\phi}\right)$$
$$- \nu\sin\theta\nabla^2 w.$$

Now

$$\nabla^2 u = \sin\theta\nabla^2 U + \cos\theta\nabla^2 V + \frac{1}{r^2}\left(2\cos\theta\frac{dU}{d\theta} - U\sin\theta\right) + \frac{\cot\theta}{r^2}U\cos\theta$$
$$- \frac{1}{r^2}\left(2\sin\theta\frac{dV}{d\theta} + V\cos\theta\right) - \frac{\cot\theta}{r^2}V\sin\theta$$

$$\nabla^2 w = \cos\theta\nabla^2 U - \sin\theta\nabla^2 V - \frac{1}{r^2}\left(2\sin\theta\frac{dU}{d\theta} + U\cos\theta\right) - \frac{\cot\theta}{r^2}U\sin\theta$$

$$-\frac{1}{r^2}\left(2\cos\theta\frac{dV}{d\theta} - V\sin\theta\right) - \frac{\cot\theta}{r^2}V\cos\theta.$$

Therefore

$$\cos\theta\nabla^2 w + \sin\theta\nabla^2 u = \nabla^2 U - \frac{U}{r^2} - \frac{2}{r^2}\frac{dV}{d\theta} - \frac{V}{r^2}\cot\theta$$

$$\cos\theta\nabla^2 u - \sin\theta\nabla^2 w = \nabla^2 V + \frac{2}{r^2}\frac{dU}{d\theta} + \frac{U}{r^2}\cot\theta - \frac{V}{r^2}.$$

Substituting the values of f_r, f_θ, and f_ϕ from (8) of § 6, we obtain

$$\left.\begin{aligned}
&\frac{\partial U}{\partial t} - \frac{V^2 + W^2}{r} = \frac{dQ}{dr} + \nu\left(\nabla^2 U - \frac{2U}{r^2} - \frac{2}{r^2}\frac{dV}{d\theta}\right.\\
&\qquad\qquad\left. - \frac{2V\cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{dW}{d\phi}\right)\\
&\frac{\partial V}{\partial t} + \frac{UV}{r} - \frac{W^2}{r}\cot\theta = \frac{1}{r}\frac{dQ}{d\theta} + \nu\left(\nabla^2 V + \frac{2}{r^2}\frac{dU}{d\theta}\right.\\
&\qquad\qquad\left. - \frac{V}{r^2\sin^2\theta} - \frac{2}{r^2}\frac{\cot\theta}{\sin\theta}\frac{dW}{d\phi}\right)\\
&\frac{\partial W}{\partial t} + \frac{UW}{r} + \frac{UV}{r}\cot\theta = \frac{1}{r\sin\theta}\frac{dQ}{d\phi} + \nu\left(\nabla^2 W - \frac{W}{r^2\sin^2\theta}\right.\\
&\qquad\qquad\left. + \frac{2}{r^2\sin\theta}\frac{dU}{d\phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{dV}{d\phi}\right)
\end{aligned}\right\} \ldots(24),$$

where $$\frac{\partial}{\partial t} = \frac{d}{dt} + U\frac{d}{dr} + \frac{V}{r}\frac{d}{d\theta} + \frac{W}{r\sin\theta}\frac{d}{d\phi}.$$

471. If the impressed forces have a potential, the equations determining the rates of change of molecular rotation in the case of a liquid, are obtained by eliminating the pressure and potential from (22); and are

$$\left.\begin{aligned}
\frac{\partial\xi}{\partial t} &= \xi\frac{du}{dx} + \eta\frac{du}{dy} + \zeta\frac{du}{dz} + \nu\nabla^2\xi\\
\frac{\partial\eta}{\partial t} &= \xi\frac{dv}{dx} + \eta\frac{dv}{dy} + \zeta\frac{dv}{dz} + \nu\nabla^2\eta\\
\frac{\partial\zeta}{\partial t} &= \xi\frac{dw}{dx} + \eta\frac{dw}{dy} + \zeta\frac{dw}{dz} + \nu\nabla^2\zeta
\end{aligned}\right\} \ldots\ldots\ldots\ldots (25),$$

472. It appears from the preceding results, that the equations of motion of a viscous fluid are of a somewhat intractable character. There are however many problems, especially those relating to the small oscillations of bodies, in which the motion is sufficiently slow to permit the terms involving the squares and products of the velocities to be neglected; in other words we may write d/dt for $\partial/\partial t$. It is also probable that there may be other problems in which the neglect of these terms may not lead to any serious error. Whenever this can safely be done, the equations of motion become considerably simplified, and when the boundaries of the fluid are plane or spherical, known methods can be employed for their solution.

Another point to be noticed is that in deducing these equations, we have assumed that the stresses due to viscosity are linear functions of the strains. This assumption is perhaps rather questionable unless the motions considered are small; and therefore the equations themselves cannot be considered to stand on a perfectly unimpeachable basis. There is however a good deal of experimental evidence to show, that they may be relied on as giving a very accurate representation of motions involving small oscillations; and we shall see in Chapter XXII, that even if the motion is not slow they give results which represent motion of a similar kind to that which actually takes place. We are therefore justified in concluding, that the preceding equations of motion give a better representation of the motion of fluids which exist in nature, than those which are derived from the supposition that the fluid is frictionless.

473. When the terms involving the squares and products of the velocities are neglected, we can deduce an important result from equations (25); for in this case they become

$$\frac{d\xi}{dt} = \nu\nabla^2\xi, \quad \frac{d\eta}{dt} = \nu\nabla^2\eta, \quad \frac{d\zeta}{dt} = \nu\nabla^2\zeta \quad\text{............(26)},$$

which shows that molecular rotation is propagated in a viscous liquid, according to the same law as heat in a conducting medium.

Impulsive Motion.

474. We shall now show that the equations of impulsive motion of a viscous liquid are the same as those of a frictionless liquid.

If we regard an impulsive force as the limit of a very large finite force which acts for a very short time τ, and if we integrate the first of (22) between the limits τ and 0, all the integrals will vanish except those in which the quantity to be integrated becomes infinite when τ vanishes; we thus obtain

$$u - u_0 + \frac{1}{\rho}\frac{d}{dx}\int_0^\tau p d\tau = 0.$$

Putting $\int_0^\tau p d\tau = P$, where P is the impulsive pressure at any point of the liquid, we obtain

$$\rho(u - u_0) + dP/dx = 0 \quad\text{.................}\quad (27),$$

with two similar equations, which are the same as those which determine the impulsive pressure at any point of a frictionless liquid.

These equations also show that it is impossible to produce any *instantaneous* change in the molecular rotation of a viscous liquid by any impulse applied to the boundary; and also that if u, v, w and $u + u', v + v', w + w'$ are the velocities just before and just after the impulse, then $u'dx + v'dy + w'dz$ a perfect differential, and is therefore derivable from a single function ϕ by differentiation; but after a sensible interval has elapsed, this quantity will no longer be a perfect differential.

Boundary Conditions.

475. We must now consider the conditions to be satisfied at the boundaries of the fluid.

At a free surface the normal stress must be constant, and the tangential stress must be zero; hence there are three equations of condition, which must be obtained from (4) and (19) by resolving the stresses upon any element of the free surface along the normal, and along two lines at right angles to it. The kinematical condition of § 12 of course always holds.

If the fluid is in contact with a fixed or moving surface, the component of the velocity perpendicular to the surface must always be equal to that of the surface. With regard to the tangential component, it is found in many cases that an indefinitely thin film of fluid adheres to the surface and moves with it. When this is the case the velocity of the fluid in contact with the surface is the same as that of the surface itself.

The experiments of Helmholtz and Piotrowski appear to indicate that in the case of many fluids, slipping may take place at the surface of a solid in contact with the fluid. When the velocity of the fluid relative to the solid is small, it is assumed that the tangential force exerted by the solid upon the fluid is in the same direction as that of the relative velocity and proportional to it; hence if u, v; u', v' be the component velocities of the fluid and solid at any point P of the solid along two lines in the tangent plane at P which are perpendicular to one another, and T, T' are the tangential stresses in these directions, the surface conditions are

$$T = \beta (u - u'), \quad T' = \beta (v - v') \quad \text{............} (28),$$

where β is *the coefficient of sliding friction.*

Prof. W. C. Unwin considers that conditions (28) hold good in the case of water, when the relative velocity is less than one inch per second. At velocities of $\frac{1}{2}$ foot per second and greater velocities, the frictional resistance is more nearly proportional to the square of the relative velocity.

Many attempts have been made to express the law of friction of a fluid in contact with a surface, in a form which is applicable to high as well as low velocities, and various empirical formulæ have been proposed. These are discussed in Prof. W. C. Unwin's Article on *Hydraulics*, in the *Encyclopædia Britannica.*

The Coefficient of Viscosity.

476. The determination of the numerical value of the coefficient of viscosity is of considerable importance, and numerous experiments have been made in recent years, especially in Germany, for the purpose of ascertaining its value. It is beyond the scope of the present treatise to attempt to discuss these experiments, and we shall therefore confine ourselves to making some general remarks

upon the subject, and giving the values of this quantity for some of the more important fluids.

The coefficient of viscosity is found to be independent of the pressure, but is dependent on the temperature.

The value of μ in C.G.S. units for the following liquids has been determined by Helmholtz and Piotrowski[1].

Liquid	μ	Temperature centigrade
Water	·014 061 22	24·5°
Alcohol	·018 917 25	24·05°
Ether	·002 496 179 5	21·6°
Carbon bisulphide	·003 365 026	21·85°

According to more recent experiments made by König[2], the values of μ for the following liquids are

Liquid	μ
Water	·014 39
Ether	·002 56
Carbon bisulphide	·003 88
Oil of Turpentine	·018 65

Shröttner found the following values of μ for glycerine at θ^0 C.

$$\mu = 42 \text{ when } \theta = 3^0,$$

$$\mu = 8 \quad ,, \quad \theta = 20^0.$$

A very elaborate series of experiments upon a variety of hydrocarbons, has been made by Pilram and Handl[3], which are discussed in a paper by Graetz[4], in which references will be found to most of the authorities on the subject.

[1] *Sitzungs. der k. k. Acad. der Wiss. zu Wien*, vol. XL. p. 607; see also *Wiss. Abhand.* vol. I. p. 172.

[2] *Wied. Ann.* 1887, p. 193.

[3] *Wien. Ber.* 1878, p. 113; 1879, p. 1; 1881, p. 11.

[4] *Wied. Ann.* 1888, p. 25.

The value of μ found by Helmholtz and Piotrowski for water at 77° Fahr. when expressed in British units of feet, pounds, pounds per square foot, feet per second is

$$\mu = \cdot 000\,001\,91.$$

For water the value of μ decreases rapidly as the temperature rises.

The following values in C.G.S. units of the coefficient of sliding friction β are given by Helmholtz and Piotrowski; it must however be confessed that these values are not of universal application, since this quantity depends not only on the particular liquid, but also on the nature of the substance with which it is in contact.

Liquid	Value of $\mu\rho/\beta$
Water	·235 34
Alcohol	·010 96
Ether	·012 43
Carbon bisulphide	·044 30

Viscosity of Gases.

477. According to the experiments of Maxwell[1], the value of μ for air at temperature θ° C. in C.G.S. units is

$$\mu = \mu_0 (1 + \cdot 003\,66\theta),$$

where μ_0 is the value of μ at 0° C.

The more recent experiments of Obermayer[2] and Holman[3] show that for air, the coefficient of viscosity increases at a less rapid rate at higher than at lower temperatures. The former has deduced from his experiments the formula

$$\mu = \mu_0 (1 + \cdot 003\,858\,5\theta - \cdot 000\,001\,05\theta^2),$$

and the latter the formula

$$\mu = \mu_0 (1 + \cdot 002\,751\theta - \cdot 000\,000\,34\theta^2).$$

[1] *Phil. Trans.* 1866.
[2] *Wien. Ber.* vol. LXXIII. p. 468 (1876).
[3] *Phil. Mag.* (5) XXI. p. 220.

The values of μ_0 for air as determined by different experimenters is

Maxwell[1]	·000 187 8
O. E. Meyer	·000 172 7
Puluj[2]	·000 179 8
Schneebeli[3]	·000 170 7
Obermayer[3]	·000 170 5
Tomlinson[4]	·000 171 55

The value of μ for air obtained by Maxwell when expressed in British units and degrees Fahr. is,

$$\mu = \cdot 000\ 000\ 025\ 6\,(461^\circ + \theta).$$

Maxwell found that damp air over water at a temperature of $21^\circ\cdot 11$ C., and a pressure of 101 millims., is *less* viscous than dry air at the same temperature by about one-sixtieth per cent. The researches of Tomlinson lead to the conclusion that at 15° C. and a pressure of 760 millims., air saturated with aqueous vapour would be *more* viscous than dry air to the extent of ·2 per cent.; and that it is not until air is under a less pressure than 350 millims., that the aqueous vapour begins to show any appreciable effect; but when the rarefaction is great, moist air is *less* viscous than dry air. See also a paper by Crookes, *On the Viscosity of Gases at High Exhaustions*[5].

Maxwell found that dry hydrogen is less viscous than air, the ratio of its viscosity to that of air being ·5156. Whence for hydrogen

$$\mu_0 = \cdot 000\ 087\ 451.$$

Also a small proportion of air mixed with hydrogen was found to produce a large increase in its viscosity, and a mixture of equal parts of hydrogen and air has a viscosity nearly equal to $\frac{15}{16}$ that of air.

The viscosity of oxygen is greater than that of air.

The experiments of Obermayer, Wiedermann and Holman

[1] *Phil. Trans.* 1866.
[2] *Phil. Mag.* vol. XXI. 1886, p. 221.
[3] *Archives des Sciences, Phy. Nat.* vol. XIV.
[4] *Phil. Trans.* 1886.
[5] *Ibid.* 1881.

have respectively led to the following formulæ for carbonic acid gas:

$$\mu = \mu_0 (1 + \cdot 003\,585\theta - \cdot 000\,001\,05\theta^2)$$

$$\mu = \mu_0 (1 + \cdot 003\,727\theta - \cdot 000\,003\,2\theta^2)$$

$$\mu = \mu_0 (1 + \cdot 003\,725\theta - \cdot 000\,002\,64\theta^2).$$

According to Maxwell the ratio of the viscosity of dry air to carbonic acid gas is about ·859, whence

$$\mu_0 = \cdot 000\,161\,310\,2.$$

Dissipation of Energy.

478. We shall now obtain an expression for the energy converted into heat.

If q be the resultant velocity, the rate of increase of kinetic energy within a closed surface S, is

$$\frac{dT}{dt} = \tfrac{1}{2}\frac{d}{dt}\iiint \rho q^2 dx\,dy\,dz$$

$$= \iiint \left(\rho q \frac{dq}{dt} + \tfrac{1}{2} q^2 \frac{d\rho}{dt}\right) dx\,dy\,dz \ldots\ldots\ldots\ldots (29).$$

Now $$q\frac{dq}{dt} = q\frac{\partial q}{\partial t} - \tfrac{1}{2}\left(u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right) q^2.$$

Also $$\frac{d\rho}{dt} = -\frac{d(\rho u)}{dx} - \frac{d(\rho v)}{dy} - \frac{d(\rho w)}{dz}.$$

Substituting in (29) and integrating the last two terms by parts, we obtain

$$\frac{dT}{dt} = \iiint \rho\left(u\frac{\partial u}{\partial t} + v\frac{\partial v}{\partial t} + w\frac{\partial w}{\partial t}\right) dx\,dy\,dz - \tfrac{1}{2}\iint \rho q^2 (lu + mv + nw)\, dS.$$

Substituting the values of $\partial u/\partial t$ &c. from the equations of motion in the first term, it becomes

$$\iiint \rho (Xu + Yv + Zw)\, dx\,dy\,dz$$

$$+ \iiint \left\{ u\left(\frac{dP}{dx} + \frac{dU}{dy} + \frac{dT}{dz}\right) + v\left(\frac{dU}{dx} + \frac{dQ}{dy} + \frac{dS}{dz}\right) \right.$$

$$\left. + w\left(\frac{dT}{dx} + \frac{dS}{dy} + \frac{dR}{dz}\right)\right\} dx\,dy\,dz,$$

and thus on integration by parts we obtain,

$$\frac{dT}{dt} = \iiint\rho\,(Xu + Yv + Zw)\,dx\,dy\,dz - \tfrac{1}{2}\iint\rho q^2\,(lu + mv + nw)\,dS$$
$$+\iint\{u\,(lP + mU + nT) + v\,(lU + mQ + nS) + w(lT + mS + nR)\}\,dS$$
$$-\iiint\{Pe + Qf + Rg + 2Sa + 2Tb + 2Uc\}\,dx\,dy\,dz.$$

The first term of this expression is equal to the rate at which kinetic energy is generated by the impressed forces which act on the fluid within the surface; the second term is the rate at which kinetic energy is introduced by the fluid crossing the boundary and bringing its kinetic energy with it; the third term is equal to $\iint(Fu + Gv + Hw)\,dS$ by (4), and therefore represents the rate of generation of energy by the stresses acting on the boundary of S; and we have to consider the last term.

Writing

$$F = -\tfrac{2}{3}\mu\,(e + f + g)^2 + 2\mu\,(e^2 + f^2 + g^2 + 2a^2 + 2b^2 + 2c^2)\ldots(30),$$

the last term is

$$\iiint p\,(e + f + g)\,dx\,dy\,dz - \iiint F\,dx\,dy\,dz.$$

The first volume integral vanishes for a liquid, and for a gas it is equal to the rate at which potential energy is converted into kinetic energy in consequence of the expansion of the gas. The last integral represents the rate at which energy is converted into heat. The function F is called by Lord Rayleigh the *dissipation function.*

It follows from (30) that when a gas expands equally in all directions $F = 0$. Whence the physical interpretation of Stokes' third assumption is, that for motion of this kind there is no dissipation of energy.

On Steady Motion.

479. When the motion of a liquid is slow, we may neglect the terms involving the squares and products of the velocities; and whenever this can be done the equations of steady motion of a liquid can be reduced to a very simple form[1].

[1] Oberbeck, *Borch.* vol. LXXXI. p. 62.

Putting $Q = -V - p/\rho$, and remembering that in steady motion $du/dt = dv/dt = dw/dt = 0$, (22) becomes

$$\left.\begin{aligned} -\frac{1}{\nu}\frac{dQ}{dx} &= \nabla^2 u = 2\left(\frac{d\eta}{dz} - \frac{d\zeta}{dy}\right) \\ -\frac{1}{\nu}\frac{dQ}{dy} &= \nabla^2 v = 2\left(\frac{d\zeta}{dx} - \frac{d\xi}{dz}\right) \\ -\frac{1}{\nu}\frac{dQ}{dz} &= \nabla^2 w = 2\left(\frac{d\xi}{dy} - \frac{d\eta}{dx}\right) \end{aligned}\right\} \quad \text{.................(31).}$$

Differentiating with respect to x, y, z, and taking account of the equation of continuity, we obtain

$$\nabla^2 Q = 0 \quad \text{..........................(32).}$$

From (26) we obtain

$$\nabla^2 \xi = 0, \quad \nabla^2 \eta = 0, \quad \nabla^2 \zeta = 0 \quad \text{.................(33),}$$

and by § 17, (26), $$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0 \quad \text{.....................(34).}$$

Also if we put

$$\left.\begin{aligned} u &= \frac{d\phi}{dx} + \frac{dN}{dy} - \frac{dM}{dz} \\ v &= \frac{d\phi}{dy} + \frac{dL}{dz} - \frac{dN}{dx} \\ w &= \frac{d\phi}{dz} + \frac{dM}{dx} - \frac{dN}{dy} \end{aligned}\right\} \quad \text{...................(35),}$$

it can be shown as in § 60, that

$$\nabla^2\phi = 0, \quad \nabla^2 L + 2\xi = 0, \quad \nabla^2 M + 2\eta = 0, \quad \nabla^2 N + 2\zeta = 0 \quad \text{...(36).}$$

480. Let f be any function of x, y, z which satisfies the equation

$$\nabla^2 f = 0,$$

and let f_0, f_1, f_2, f_3 be four new functions of x, y, z which satisfy the equations

$$f_0 = f + x\frac{df}{dx} + y\frac{df}{dy} + z\frac{df}{dz} \quad \text{...............(37),}$$

$$\left.\begin{aligned} f_1 &= z\frac{df}{dy} - y\frac{df}{dz} \\ f_2 &= x\frac{df}{dz} - z\frac{df}{dx} \\ f_3 &= y\frac{df}{dx} - x\frac{df}{dz} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38).$$

Then it can easily be shown that

$$\nabla^2 f_0 = \nabla^2 f_1 = \nabla^2 f_2 = \nabla^2 f_3 = 0 \ldots\ldots\ldots\ldots\ldots (39),$$

$$\frac{df_1}{dx} + \frac{df_2}{dy} + \frac{df_3}{dz} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (40),$$

$$\left.\begin{aligned} \frac{df_3}{dy} - \frac{df_2}{dz} &= \frac{df_0}{dx} \\ \frac{df_1}{dz} - \frac{df_3}{dx} &= \frac{df_0}{dy} \\ \frac{df_2}{dx} - \frac{df_1}{dy} &= \frac{df_0}{dz} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41).$$

Comparing equations (39), (40) and (41) with (32), (33), (34) and (31), we see that the same equations are satisfied by f_0, f_1, f_2 and f_3 as are satisfied by Q/ν, ξ, η and ζ; hence we may put

$$Q/\nu = f_0,\; 2\xi = f_1,\; 2\eta = f_2,\; 2\zeta = f_3 \ldots\ldots\ldots\ldots (42).$$

481. In the next place we shall show that we may put

$$\left.\begin{aligned} L &= z\frac{dF}{dy} - y\frac{dF}{dz} \\ M &= x\frac{dF}{dz} - z\frac{dF}{dx} \\ N &= y\frac{dF}{dx} - x\frac{dF}{dy} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43),$$

where F is a function to be determined. For substituting these values in the first of (35) we obtain

$$u = \frac{d\phi}{dx} + \frac{d}{dx}\left(F + x\frac{dF}{dx} + y\frac{dF}{dy} + z\frac{dF}{dz}\right) - x\nabla^2 F \ldots (44),$$

with similar expressions for v and w; and if we differentiate the right hand sides of (44) with respect to x, y, z respectively, it will

be found that these values of u, v and w, and therefore of L, M, N satisfy the equation of continuity.

From equations (36) and (43) we obtain

$$-2\xi = \nabla^2 L = z\nabla^2 \frac{dF}{dy} - y\nabla^2 \frac{dF}{dz},$$

whence by (38) and (42)

$$\nabla^2 F = -f,$$

therefore

$$\nabla^2\nabla^2 F = 0 \qquad \text{.......................(45)}.$$

We have thus reduced every problem of steady motion to the determination of two functions ϕ and F which respectively satisfy the equations

$$\nabla^2\phi = 0, \quad \nabla^2\nabla^2 F = 0 \qquad \text{..................(46)}.$$

482. We shall conclude this chapter with two general propositions.

When the motion is steady and there are no impressed forces and the squares and products of the velocities are neglected, the sum of the surface integrals of each of the components of stress parallel to the axes, taken over each of the bounding surfaces is zero.

From (5) we obtain

$$\frac{dP}{dx} + \frac{dU}{dy} + \frac{dT}{dz} = 0,$$

whence

$$\iiint \left(\frac{dP}{dx} + \frac{dU}{dy} + \frac{dT}{dz}\right) dx\,dy\,dz = 0,$$

where the volume integral extends throughout the fluid. Integrating by parts we obtain

$$0 = \iint (Pl + Um + Tn)\, dS = \iint F dS$$

by (4); where the surface integral is to be taken over each of the bounding surfaces.

483. *When the motion of a liquid is steady and the squares and products of the velocities are neglected, and no slipping takes place at the surfaces of solids in contact with it, the loss of energy is less than it would be if the liquid had any other motion consistent with the boundary conditions*[1].

[1] Helmholtz, *Wiss. Abhand*, vol. I. p. 223.

The loss of energy per unit of volume in the case of a liquid, is given by the last terms of the dissipation function F, we have therefore to find the conditions that $\iiint F dxdydz$ may be a minimum, subject to the condition

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

Hence if λ is an undetermined function of x, y, z, we must have

$$\delta \iiint \{F + \lambda (u_x + v_y + w_z)\} \, dxdydz = 0.$$

Taking the variation, we obtain

$$2\mu \iiint \{2u_x \delta u_x + 2v_y \delta v_y + 2w_z \delta w_z + (v_z + w_y)(\delta v_z + \delta w_y) + (w_x + u_z)(\delta w_x + \delta u_z)$$
$$+ (u_y + v_x)(\delta u_y + \delta v_x)\} \, dxdydz + \iiint \lambda (\delta u_x + \delta v_y + \delta w_z) \, dxdydz = 0.$$

Integrating each term by parts we obtain

$$2\mu \iint \{2lu_x + m(v_x + u_y) + n(w_x + u_z)\} \, \delta u dS + \text{two similar terms}$$
$$- 2\mu \iiint (\nabla^2 u \delta u + \nabla^2 v \delta v + \nabla^2 w \delta w) \, dxdydz,$$
$$+ \iint \lambda (l\delta u + m\delta v + n\delta w) \, dS - \iiint \left(\frac{d\lambda}{dx} \delta u + \frac{d\lambda}{dy} \delta v + \frac{d\lambda}{dz} \delta w\right) dxdydz.$$

In order that the volume integrals may vanish, we must have

$$2\mu \nabla^2 u + \frac{d\lambda}{dx} = 0, \quad 2\mu \nabla^2 v + \frac{d\lambda}{dy} = 0, \quad 2\mu \nabla^2 w + \frac{d\lambda}{dz} = 0.$$

Comparing these equations with (31) we see that

$$\lambda = 2Q\rho = -2(V\rho + p).$$

Also since there is no slipping, and the boundary conditions are assumed to be unaltered, δu, δv and δw are each zero at the boundaries, whence the surface integrals vanish.

EXAMPLES.

1. When the motion of a liquid is in two dimensions, and the squares and products of the velocities are neglected, prove that the equations of motion are

$$r\frac{dQ}{dr}+\frac{d\chi}{d\theta}=0,\quad \frac{dQ}{d\theta}-r\frac{d\chi}{dr}=0,$$

where

$$\chi=\nu\left(\frac{d^2\psi}{dr^2}+\frac{1}{r}\frac{d\psi}{dr}+\frac{1}{r^2}\frac{d^2\psi}{d\theta^2}\right)-\frac{d\psi}{dt},$$

and ψ is Earnshaw's current function.

2. When the motion of a liquid is in two dimensions and the squares and products of the velocity are *not* neglected, prove that ψ satisfies the equation

$$\left(\nu\nabla^2-\frac{d}{dt}\right)\nabla^2\psi=\left(u\frac{d}{dx}+v\frac{d}{dy}\right)\nabla^2\psi.$$

3. Viscous liquid is confined between fixed walls at which there is no slipping; prove that the rate at which energy is diminishing is

$$4\mu\iiint\omega^2 dxdydz,$$

where ω is the molecular rotation.

4. If T be the kinetic energy of a viscous liquid which is contained within a closed surface S, prove that

$$\frac{dT}{dt}=-4\mu\iiint\omega^2 dxdydz+2\mu\iint q\omega\sin\chi\sin\theta dS,$$

where q is the resultant velocity, ω the molecular rotation, χ the angle between the directions of q and the instantaneous axis of rotation, and θ is the inclination of the normal to the plane containing the latter axis and the direction of q.

5. Prove that the values of the six component stresses in polar coordinates are

$$P=-p-\tfrac{2}{3}\mu\delta+2\mu\frac{du}{dr},\quad Q=-p-\tfrac{2}{3}\mu\delta+2\mu\left(\frac{1}{r}\frac{dv}{d\theta}+\frac{u}{r}\right),$$

$$R=-p-\tfrac{2}{3}\mu\delta+2\mu\left(\frac{1}{r\sin\theta}\frac{dw}{d\phi}+\frac{u}{r}+\frac{v}{r}\cot\theta\right),$$

$$S=\mu\left(\frac{1}{r\sin\theta}\frac{dv}{d\phi}+\frac{1}{r}\frac{dw}{d\theta}-\frac{w}{r}\cot\theta\right),\quad T=\mu\left(\frac{1}{r\sin\theta}\frac{du}{d\phi}+\frac{dw}{dr}-\frac{w}{r}\right),$$

$$U=\mu\left(\frac{1}{r}\frac{du}{d\theta}+\frac{dv}{dr}-\frac{v}{r}\right),$$

where

$$\delta=\frac{du}{dr}+\frac{1}{r}\frac{dv}{d\theta}+\frac{1}{r\sin\theta}\frac{dw}{d\phi}+\frac{2u}{r}+\frac{v}{r}\cot\theta.$$

6. Prove also that the values of the same quantities in cylindrical coordinates are,

$$P=-p-\tfrac{2}{3}\mu\delta+2\mu\frac{du}{d\varpi},\quad Q=-p-\tfrac{2}{3}\mu\delta+2\mu\left(\frac{1}{\varpi}\frac{dv}{d\theta}+\frac{u}{\varpi}\right),$$

$$R=-p-\tfrac{2}{3}\mu\delta+2\mu\frac{dw}{dz},$$

$$S=\mu\left(\frac{1}{\varpi}\frac{dw}{d\theta}+\frac{dv}{dz}\right),\quad T=\mu\left(\frac{du}{dz}+\frac{dw}{d\varpi}\right),\quad U=\mu\left(\frac{dv}{d\varpi}-\frac{v}{\varpi}+\frac{1}{\varpi}\frac{du}{d\theta}\right).$$

7. When the motion of a liquid is symmetrical with respect to the axis of z, prove that Stokes' current function satisfies the equation

$$\left(\nu D-\frac{d}{dt}\right)D\psi=\left(u\frac{d}{d\varpi}+w\frac{d}{dz}-\frac{2u}{\varpi}\right)D\psi,$$

where $$D=\frac{d^2}{dz^2}+\frac{d^2}{d\varpi^2}-\frac{1}{\varpi}\frac{d}{d\varpi}.$$

CHAPTER XXI.

ON THE STEADY MOTION AND SMALL OSCILLATIONS OF SOLID BODIES IN A VISCOUS LIQUID.

484. THE first problems relating to the motion of solid bodies in a viscous liquid were solved by Prof. Stokes[1], who in 1850 obtained the solution in the case of a sphere which is constrained to move with uniform velocity in a straight line, after a sufficient time has elapsed for the motion to have become steady, and also in the case of spherical and cylindrical pendulums, which are performing small oscillations in a straight line. The torsional oscillations of spheres and cylinders form the subject of a joint memoir by Helmholtz and Piotrowski[2], and Oberbeck[3] has obtained the solution in the case of the steady motion of an ellipsoid which moves parallel to an axis. We shall devote the present Chapter to the consideration of these investigations.

Motion of a Sphere.

485. Let us suppose that a sphere of radius a is moving along a straight line which we shall choose for the axis of z, and that the initial motion of the liquid is symmetrical with respect to this axis; then it is evident that the subsequent motion will also be symmetrical with respect to this axis, and therefore the motion of the liquid can be determined by means of Stokes' current function.

[1] "On the Effect of the Internal Friction of Fluids on the Motion of Pendulums," *Trans. Camb. Phil. Soc.* vol. IX.

[2] *Wissenschaft. Abhand.* vol. I. p. 172.

[3] *Borch.* vol. LXXXI. p. 62.

Hence if w and u be the component velocities along and perpendicular to the axis of z,

$$u = -\frac{1}{\varpi}\frac{d\psi}{dz}, \quad w = \frac{1}{\varpi}\frac{d\psi}{d\varpi} \quad \text{.................. (1).}$$

The equations of motion are determined by (23) of § 470; and if we neglect the terms involving the squares and products of the velocity, and remember that none of the quantities are functions of θ, we obtain

$$\frac{du}{dt} = \frac{dQ}{d\varpi} + \nu\left(\nabla^2 u - \frac{u}{\varpi^2}\right) \quad \text{.................. (2),}$$

$$\frac{dw}{dt} = \frac{dQ}{dz} + \nu\nabla^2 w \quad \text{.......................... (3),}$$

and the equation of continuity is

$$\frac{dw}{dz} + \frac{du}{d\varpi} + \frac{u}{\varpi} = 0 \text{........................(4).}$$

Equations (2) and (3) have been formed on the supposition that the origin is fixed; let us now suppose that the motion is referred to the centre of the sphere as origin, which is supposed to be moving along the axis of z with velocity V, and let ζ be its distance from a fixed point. If (z, ϖ) be the coordinates of a point referred to the centre of the sphere as origin,

$$w = f(z + \zeta, \varpi, t),$$

therefore

$$\frac{dw}{dt} = \frac{df}{dt} + V\frac{df}{d\zeta},$$

the second term on the right-hand side is of the same order as the square of the velocity, and must therefore be omitted; hence on the supposition that such terms can be neglected, (2) and (3) hold good whether the origin is fixed or in motion.

Let

$$D = \frac{d^2}{dz^2} + \frac{d^2}{d\varpi^2} - \frac{1}{\varpi}\frac{d}{d\varpi} \text{.....................(5),}$$

then if R be any function of z and ϖ

$$\nabla^2\left(\frac{R}{\varpi}\right) = \frac{1}{\varpi}DR + \frac{R}{\varpi^3},$$

and

$$\frac{d}{d\varpi}(DR) = D\left(\frac{dR}{d\varpi}\right) + \frac{1}{\varpi^2}\frac{dR}{d\varpi};$$

whence (2) and (3) may be written

$$\left.\begin{aligned} \frac{dQ}{dz} &= \frac{1}{\varpi}\frac{d}{d\varpi}\left(\frac{d\psi}{dt} - \nu D\psi\right) \\ -\frac{dQ}{d\varpi} &= \frac{1}{\varpi}\frac{d}{dz}\left(\frac{d\psi}{dt} - \nu D\psi\right) \end{aligned}\right\} \quad \text{.................(6).}$$

Eliminating Q, the equation for determining ψ is

$$D\left(D - \frac{1}{\nu}\frac{d}{dt}\right)\psi = 0 \quad \text{.................... (7).}$$

The solution of this equation is

$$\psi = \psi_1 + \psi_2 \quad \text{.........................(8),}$$

where ψ_1 and ψ_2 respectively satisfy the equations

$$D\psi_1 = 0 \quad \text{.............................(9),}$$

$$\left(D - \frac{1}{\nu}\frac{d}{dt}\right)\psi_2 = 0 \quad \text{.................... (10).}$$

Multiplying (6) by dz, $d\varpi$, subtracting and taking account of (9) and (10), we obtain

$$-dQ = \frac{1}{\varpi}\frac{d}{dt}\left(\frac{d\psi_1}{dz}\,d\varpi - \frac{d\psi_1}{d\varpi}\,dz\right) \quad \text{...........(11).}$$

Equation (9) shows that the right-hand side of (11) is a perfect differential.

486. We shall now transform these equations to polar coordinates r and θ.

Let R and Θ be the velocities along and perpendicular to the direction of r, then

$$R = \frac{1}{r^2 \sin\theta}\frac{d\psi}{d\theta}, \quad \Theta = -\frac{1}{r\sin\theta}\frac{d\psi}{dr} \quad \text{...........(12),}$$

and

$$D = \frac{d^2}{dr^2} + \frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\operatorname{cosec}\theta\,\frac{d}{d\theta}\right) \quad \text{.............(13),}$$

whence (11) becomes

$$-dQ = \frac{1}{\sin\theta}\frac{d}{dt}\left(\frac{d\psi_1}{dr}\,d\theta - \frac{1}{r^2}\frac{d\psi_1}{d\theta}\,dr\right) \quad \text{......... (14).}$$

487. We must now consider the boundary conditions. If we suppose that there is no slipping at the surface of the sphere, the boundary conditions are

$$R = V\cos\theta, \quad \Theta = -V\sin\theta \quad \text{..............(15),}$$

where V is the velocity of the sphere. By (12) these are equivalent to

$$\frac{d\psi}{d\theta} = Va^2 \sin\theta\cos\theta, \quad \frac{d\psi}{dr} = Va\sin^2\theta \text{..........}(16).$$

If there is slipping, let U be the tangential stress exerted by the liquid on the sphere in the direction of θ; then it follows from § 475 that we must substitute for the second of (16)

$$\beta(\Theta + V\sin\theta) = \mu U.$$

By means of the transformation formulae of § 18, it can be shown that

$$U = \mu\left(\frac{1}{r}\frac{dR}{d\theta} + \frac{d\Theta}{dr} - \frac{\Theta}{r}\right),$$

whence the condition becomes

$$\beta(\Theta + V\sin\theta) = \mu\left(\frac{1}{a}\frac{dR}{d\theta} + \frac{d\Theta}{dr} - \frac{\Theta}{a}\right) \text{.........}(17),$$

in which r is to be put equal to a after differentiation.

If the liquid extends to infinity and is at rest there, R and Θ must each vanish when $r = \infty$.

Equations (9), (10) and (11) together with (16) or (17) contain the complete solution of every problem relating to the rectilinear motion of a sphere in a viscous liquid of unlimited extent, which is either initially at rest, or whose initial motion is symmetrical with respect to the line along which the sphere moves. When the motion is neither of an oscillatory character nor steady, the difficulty of integrating these equations is considerable, but the solution as will be shown in the next chapter, can in certain cases be effected by means of definite integrals.

Motion of a Spherical Pendulum[1].

488. In order to apply the preceding results to the motion of a spherical pendulum, which is performing small oscillations along a straight line, we shall assume that the time enters into ψ in the form of the factor $\epsilon^{\lambda^2 \nu t}$, where λ is at present undetermined; then (9) and (10) become

$$D\psi_1' = 0, \quad (D - \lambda^2)\psi_2' = 0 \text{.................}(18),$$

[1] Stokes, *Trans. Camb. Phil. Soc.* vol. IX.; see also O. E. Meyer, *Borch.* vol. LXXIII. p. 31.

where ψ_1', ψ_2' are functions of r and θ only. Dropping the accents for the present, and putting $\mu = \cos\theta$, we can satisfy (18) by assuming

$$\psi_1 = R_n Q_n, \quad \psi_2 = R_n' Q_n$$

where R_n, R_n' are functions of r alone, and

$$Q_n = (1-\mu^2)\, dP_n/d\mu,$$

where P_n is a zonal harmonic of degree n. The equations for determining R_n, R_n' are

$$\frac{d^2 R_n}{dr^2} - n(n+1) R_n r^{-2} = 0 \qquad (19),$$

$$\frac{d^2 R_n'}{dr^2} - n(n+1) R_n' r^{-2} - \lambda^2 R_n' = 0 \qquad (20).$$

The solution of (19) is

$$R_n = Ar^{-n} + Br^{n+1}.$$

Equation (20) is discussed in Forsyth's *Differential Equations* § 112 and § 139 Example 4, and it is shown that the solution can be expressed either in the form

$$R_n' = r^{n+1}\left(\frac{1}{r}\frac{d}{dr}\right)^n \left\{\frac{C\epsilon^{\lambda r} + D\epsilon^{-\lambda r}}{r}\right\},$$

or

$$R_n' = Cr^{n+1}\int_{-\lambda}^{\lambda} \epsilon^{ru}(\lambda^2 - u^2)^n\, du + Dr^{n+1}\int_{\lambda}^{\infty} \epsilon^{-ru}(u^2-\lambda^2)^n\, du.$$

489. It will not however be necessary to consider the general solutions of (19) and (20), since the surface condition (16) shows that θ must enter into ψ in the form of the factor $\sin^2\theta$, whence $n = 1$, and

$$R = A/r + Br^2.$$

To integrate (20) when $n = 1$, put $R' = r\,dw/dr$, and the equation becomes

$$\frac{d^3 w}{dr^3} + \frac{2}{r}\frac{d^2 w}{dr^2} - \frac{2}{r^2}\frac{dw}{dr} - \lambda^2 \frac{dw}{dr} = 0.$$

Integrating we obtain

$$\frac{d^2}{dr^2}(wr) - \lambda^2 wr = 0,$$

the solution of which is

$$w = (D\epsilon^{\lambda r} - C\epsilon^{-\lambda r})/\lambda r,$$

whence
$$R' = r\frac{d}{dr}\{(\lambda r)^{-1}(D\epsilon^{\lambda r} - C\epsilon^{-\lambda r})\}$$
$$= C\epsilon^{-\lambda r}\left(1 + \frac{1}{\lambda r}\right) + D\epsilon^{\lambda r}\left(1 - \frac{1}{\lambda r}\right).$$

Since the velocity must vanish at infinity, $B = D = 0$, whence
$$\psi = \epsilon^{\lambda^2 \nu t}\left\{\frac{A}{r} + C\left(1 + \frac{1}{\lambda r}\right)\epsilon^{-\lambda r}\right\}\sin^2\theta \text{............(21)}.$$

In order to satisfy (16) we must have
$$\dot{\zeta} = V = c\epsilon^{\lambda^2 \nu t},$$
where ζ is the displacement of the centre of the sphere, and c is a constant, whence
$$\zeta = c\epsilon^{\lambda^2 \nu t}/\lambda^2\nu = V/\lambda^2\nu.$$

Also substituting the values of $d\psi/dr$ and $d\psi/d\theta$ from (21) in (16), we obtain
$$A = \tfrac{1}{2}a^3 c + \frac{3a^2 c}{2\lambda}\left(1 + \frac{1}{\lambda a}\right),$$
$$C = -\frac{3ac}{2\lambda}\epsilon^{\lambda a},$$
whence
$$\psi = \tfrac{1}{2}Va^2\left\{\left(1 + \frac{3}{\lambda a} + \frac{3}{\lambda^2 a^2}\right)\frac{a}{r} - \frac{3}{\lambda a}\left(1 + \frac{1}{\lambda r}\right)\epsilon^{-\lambda(r-a)}\right\}\sin^2\theta \text{...(22)}.$$

490. We must now calculate the resistance exerted by the liquid upon the sphere.

Let P be the normal and U the tangential stresses measured in the r and θ directions; the formulae of transformation of Chapter I., give
$$\left.\begin{aligned} P &= -p + 2\mu\frac{dR}{dr} \\ U &= \mu\left(\frac{d\Theta}{dr} + \frac{1}{r}\frac{dR}{d\theta} - \frac{\Theta}{r}\right)\end{aligned}\right\}\text{..................(23)}.$$

Hence if Z be the resistance experienced by the sphere,
$$Z = 2\pi a^2\int_0^\pi(-P\cos\theta + U\sin\theta)\sin\theta d\theta\text{........(24)}.$$

Now by (12) and (16),
$$\frac{dR}{dr} = \frac{\operatorname{cosec}\theta}{r^2}\left(\frac{d^2\psi}{drd\theta} - \frac{2}{r}\frac{d\psi}{d\theta}\right) = 0,$$
at the surface; also

$$\frac{1}{r}\frac{dR}{d\theta} = \frac{1}{r^3}\frac{d}{d\theta}\left(\operatorname{cosec}\theta\frac{d\psi}{d\theta}\right)$$

$$= \frac{1}{a^3}\frac{d}{d\theta}(Va^2\cos\theta) = -\frac{V}{a}\sin\theta = \left(\frac{\Theta}{r}\right)_a,$$

at the surface; and

$$\frac{d\Theta}{dr} = -\frac{1}{r\sin\theta}\left(\frac{d^2\psi}{dr^2} - \frac{1}{r}\frac{d\psi}{dr}\right)$$

$$= -\frac{1}{r\sin\theta}\left\{\frac{1}{\nu}\frac{d\psi_2}{dt} - \frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\operatorname{cosec}\theta\frac{d\psi}{d\theta}\right) - \frac{1}{r}\frac{d\psi}{dr}\right\}$$

by (10). Hence by (16),

$$\frac{d\Theta}{dr} = -\frac{1}{\nu a\sin\theta}\frac{d\psi_2}{dt},$$

at the surface; therefore

$$Z = 2\pi a\int_0^\pi \left(pa\cos\theta - \rho\frac{d\psi_2}{dt}\right)\sin\theta d\theta \text{.........(25).}$$

Also $\displaystyle\int_0^\pi p\cos\theta\sin\theta d\theta = -\tfrac{1}{2}\int_0^\pi \sin^2\theta\frac{dp}{d\theta}d\theta$;

but $Q = -p/\rho + gr\sin\theta$, therefore by (14) at the surface

$$\frac{dp}{d\theta} = \rho\operatorname{cosec}\theta\frac{d^2\psi_1}{drdt} + g\rho a\cos\theta,$$

therefore

$$\int_0^\pi p\cos\theta\sin\theta d\theta = -\tfrac{1}{2}\rho\int_0^\pi \left(\frac{d^2\psi_1}{drdt} + ga\sin\theta\cos\theta\right)\sin\theta d\theta,$$

accordingly $\displaystyle Z = -\pi\rho a\frac{d}{dt}\int_0^\pi \left\{a\left(\frac{d\psi_1}{dr}\right)_a + 2\psi_2\right\}\sin\theta\, d\theta$ ....(26).

Now $$\psi_1 = \frac{Va^3}{2r}\left(1 + \frac{3}{\lambda a} + \frac{3}{\lambda^2a^2}\right)\sin^2\theta.$$

Therefore $$\left(\frac{d\psi_1}{dr}\right)_a = -\tfrac{1}{2}Va\left(1 + \frac{3}{\lambda a} + \frac{3}{\lambda^2a^2}\right)\sin^2\theta,$$

and $$\psi_2 = -\frac{3Va}{2\lambda}\left(1 + \frac{1}{\lambda a}\right)\sin^2\theta.$$

Whence $$Z = \tfrac{1}{2}\pi\rho a^3\frac{d}{dt}\int_0^\pi V\left(1 + \frac{9}{\lambda a} + \frac{9}{\lambda^2a^2}\right)\sin^3\theta\, d\theta$$

$$= \tfrac{1}{2}m\left(1 + \frac{9}{\lambda a} + \frac{9}{\lambda^2a^2}\right)\frac{dV}{dt},$$

where m is the mass of the liquid displaced by the sphere.

491. Let us now suppose that the sphere is constrained to perform small oscillations of period τ. In this case we must have $\lambda^2\nu = \iota n$, where $\tau = 2\pi/n$. Putting $k=(n/2\nu)^{\frac{1}{2}}$, we obtain $\lambda=k(1+\iota)$ and

$$Z = \tfrac{1}{2}m\left[\left\{1+\frac{9}{2ka}(1-\iota)\right\}\frac{dV}{dt}+\frac{9n}{2k^2a^2}V\right]$$

$$= \tfrac{1}{2}m\left\{\left(1+\frac{9}{2ka}\right)\frac{dV}{dt}+\frac{9n}{2ka}\left(1+\frac{1}{ka}\right)V\right\} \text{.........(27)},$$

since $\iota dV/dt = -nV$.

The effect of the first term is simply to produce an apparent increase in the inertia of the sphere; the second term would produce a gradual diminution of the arc of oscillation if the sphere were left free. Now ν is a small quantity, whence k is a large quantity, hence the effect produced by the second term is small, and is almost insensible during the period of a single oscillation. We may therefore employ the preceding value of Z to obtain the correction due to viscosity in the case of a free pendulum oscillating in a liquid.

If l be the length of the pendulum, and if K, K' denote the values of the coefficients of dV/dt and V in the expression for Z; the equation of motion of the sphere will be

$$(Ml+K)\ddot{\theta}+K'\dot{\theta}+(M-m)g\theta=0,$$

the solution of which is

$$\theta = A\epsilon^{-\delta t}\sin(pt+\alpha),$$

where $$\delta = \frac{K'}{2(Ml+K)},\quad p=\frac{\{4(M-m)(Ml+K)g-K'^2\}^{\frac{1}{2}}}{2(Ml+K)}$$

The modulus of decay, that is the time which must elapse before the amplitude falls to e^{-1} of its original value, is therefore equal to $2(Ml+K)/K'$.

Torsional Oscillations of a Sphere[1].

492. We shall now investigate the motion of a sphere which is either filled with liquid or surrounded with liquid, and which is oscillating by means of a torsion fibre.

[1] Helmholtz and Piotrowski, *Wissenschaft. Abhand.* vol. I. p. 172.

Let ω be the angular velocity of the sphere, w the component velocity of the liquid in the plane perpendicular to the axis of oscillation, T the tangential stress which opposes the motion of the sphere at a point whose co-latitude is θ. The surface condition when slipping is supposed to exist is

$$\beta(w - a\omega \sin\theta) = T.$$

By (26) and (30) of § 18,

$$T = \mu\left(\frac{dw}{dr} - \frac{w}{r}\right),$$

whence

$$\beta(w - a\omega\sin\theta) = \mu\left(\frac{dw}{dr} - \frac{w}{r}\right) \quad\ldots\ldots\ldots\ldots (28).$$

Since the resistance experienced by the sphere depends solely on the component velocity w, it will be unnecessary to consider the other two components. On account of the symmetry of the motion all the quantities will be functions of (r, θ, t), whence by (24) of § 470 the equation for w will be

$$\frac{dw}{dt} = \nu\left(\nabla^2 w - \frac{w}{r^2\sin^2\theta}\right) \quad\ldots\ldots\ldots\ldots (29),$$

the squares and products of the velocities being neglected. Equation (29) will be satisfied by putting $w = W\sin\theta$, where W is a function of r and t only, which satisfies the equation,

$$\frac{1}{\nu}\frac{dW}{dt} = \frac{d^2W}{dr^2} + \frac{2}{r}\frac{dW}{dr} - \frac{2W}{r^2} \quad\ldots\ldots\ldots\ldots (30).$$

Let Ω be the angular velocity of the liquid, then $W = \Omega r$ and (30) becomes,

$$\frac{d\Omega}{dt} = \nu\left(\frac{d^2\Omega}{dr^2} + \frac{4}{r}\frac{d\Omega}{dr}\right) \quad\ldots\ldots\ldots\ldots (31).$$

This is the general equation for determining the angular velocity of a viscous liquid bounded externally or internally by a sphere which is rotating about a fixed diameter.

493. In considering the oscillations of a sphere filled with liquid, we may put

$$\Omega = \phi\epsilon^{\lambda^2\nu t},$$

whence

$$\frac{d^2\phi}{dr^2} + \frac{4}{r}\frac{d\phi}{dr} = \lambda^2\phi \quad\ldots\ldots\ldots\ldots (32),$$

the solution of which is

$$\phi = \frac{1}{r}\frac{d}{dr}\left(\frac{\epsilon^{\pm\lambda r}}{r}\right),$$

whence
$$\phi = \frac{A}{2r^2}\left(\lambda - \frac{1}{r}\right)\epsilon^{\lambda r} + \frac{B}{2r^2}\left(\lambda + \frac{1}{r}\right)\epsilon^{-\lambda r} \quad \text{.........(33).}$$

Since Ω must not be infinite when $r = 0$, we must have $A = B$, whence

$$\Omega = A\epsilon^{\lambda^2 \nu t}\left(\frac{\lambda}{r^2}\cosh \lambda r - \frac{1}{r^3}\sinh \lambda r\right) \quad \text{...........(34).}$$

Putting $k = \nu\rho/\beta$, (28) becomes

$$\Omega - \omega = k\left(\frac{d\Omega}{dr}\right)_a \quad \text{.......................(35).}$$

Now ω must be of the form $c\epsilon^{\lambda^2\nu t}$, where c is a constant; whence substituting the value of Ω from (34) we obtain

$c/A = \lambda a^{-2}(1 + 3k/a)\cosh \lambda a - a^{-2}(\lambda^2 k + a^{-1} + 3k/a^2)\sinh \lambda a$...(36),

which determines A.

The couple which the liquid exerts on the sphere measured in the direction of its motion is

$$G = -2\pi a^3 \int_0^\pi T \sin^3 \theta d\theta = -\tfrac{8}{3}\pi\nu\rho a^4 \left(\frac{d\Omega}{dr}\right)_a,$$

$$= -\tfrac{8}{3}\pi\beta a^4(\Omega - \omega) \quad \text{..............................(37),}$$

by (35).

In order to complete the solution we may proceed as in § 491. First suppose the sphere to perform small oscillations whose period is $2\pi/n$, then $\lambda^2\nu = \iota n$, and

$$\dot{\omega} = \iota n \omega, \quad \lambda = (n/2\nu)^{\frac{1}{2}}(1 + \iota) \quad \text{..............(38).}$$

By means of (36) and (38) the imaginary quantity ι can be eliminated, and the value of G expressed in a real form as a function of $\dot{\omega}$ and ω; and since the motion is supposed to be slow we may neglect squares and products of $\dot{\omega}$ and ω. Having obtained the value of G in a real form, the equation of motion which will be of the form

$$I\ddot{\theta} + \gamma\dot{\theta} + \delta\theta = 0,$$

can be written down and integrated.

If the sphere is surrounded with liquid we must put $A = 0$ in (33), because ϕ must not be infinite when $r = \infty$. If there is no slipping $\beta = \infty$ and therefore $k = 0$,

Steady Motion of a Sphere[1].

494. We shall first consider the steady motion of a sphere which is moving along a straight line, when slipping takes place; in order to pass to the case of no slipping, we must put $\beta = \infty$ in our results.

When the motion is steady $d\psi/dt = 0$, and (7) becomes

$$D^2\psi = 0 \quad\text{..........................(39).}$$

Let $\psi = \phi(r)\sin^2\theta$, then (39) becomes

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)^2 \phi = 0,$$

whence
$$\frac{d^2\phi}{dr^2} - \frac{2\phi}{r^2} = \frac{A'}{r} + B'r^2.$$

Integrating again and changing the constants, we obtain

$$\phi = A/r + Br + Cr^2 + Dr^4 \quad\text{.................(40).}$$

Since R and Θ vanish at infinity, it follows that $C = 0$, $D = 0$, and

$$\psi = (A/r + Br)\sin^2\theta.$$

The first of (16) gives

$$A + Ba^2 = \tfrac{1}{2}Va^3,$$

and (17) gives
$$A(1 + 6\mu/\beta a) - Ba^2 = -Va^3,$$

whence

$$A = -\tfrac{1}{4}Va^3/(1 + 3\mu/\beta a), \quad B = \tfrac{3}{4}Va(1 + 2\mu/\beta a)/(1 + 3\mu/\beta a).$$

We thus obtain

$$\psi = \tfrac{1}{4}Va^2\sin^2\theta\left\{3\left(1 + \frac{2\mu}{\beta a}\right)\frac{r}{a} - \frac{a}{r}\right\}\left(1 + \frac{3\mu}{\beta a}\right)^{-1} \quad\text{... (41).}$$

If there is no slipping $\beta = \infty$, and the preceding equation becomes

$$\psi = \tfrac{1}{4}Va^2\left(\frac{3r}{a} - \frac{a}{r}\right)\sin^2\theta \quad\text{.................(42),}$$

which is Prof. Stokes' result.

495. The value of the force which must be applied to the sphere in order to maintain the motion, may be obtained either by calculating the resultant force exerted by the liquid upon the

[1] Stokes, *Trans. Camb. Phil. Soc.* vol. IX.; see also Lamb, *Motion of Fluids*, §§ 184—185.

sphere, or by means of the dissipation function. If we employ the first method, and put $u = D\psi$, we obtain from (6)

$$\frac{dp}{\rho} = \frac{\nu}{\varpi}\left(\frac{du}{d\varpi}\,dz - \frac{du}{dz}\,d\varpi\right),$$

or $$dp = \frac{\mu}{r\sin\theta}\left(\frac{1}{r}\frac{du}{d\theta}\,dr - r\frac{du}{dr}\,d\theta\right).$$

Now $$D\psi = -2Br^{-2}\sin^2\theta,$$

whence $$dp = 2B\mu d\,(r^{-2}\cos\theta),$$

and therefore $$p = \Pi + 2B\mu r^{-2}\cos\theta \quad\text{.....................(43)},$$

and we obtain from (23),

$$P = -p + 2\mu\frac{dR}{dr},$$

$$= -\Pi - \mu\cos\theta\,(12A/a^4 + 6B/a^2),$$

also $$U = -6A\mu r^{-4}\sin\theta,$$

and therefore from (24)

$$Z = 2\pi\mu\int_0^\pi \left\{\left(\frac{12A}{a^2} + 6B\right)\cos^2\theta\sin\theta - \frac{6A}{a^2}\sin^3\theta\right\}d\theta,$$

$$= 8\pi\mu B,$$

$$= 6V\pi\mu a\,(1 + 2\mu/\beta a)/(1 + 3\mu/\beta a) \quad\text{.....................(44)}.$$

If in (44) we put β respectively equal to infinity and zero, we see that Z must lie between the values $6V\pi\mu a$ and $4V\pi\mu a$.

If a solid of density σ is descending in a viscous liquid of density ρ under the action of gravity, the force in the direction of its motion is $\frac{4}{3}\pi a^3 g\,(\sigma - \rho)$. If therefore the sphere descend from rest, the velocity will not continue to increase indefinitely, but will tend towards a limiting value which is determined by the equation

$$\tfrac{4}{3}\pi a^3 g\,(\sigma - \rho) = 6V\pi\mu a\,(1 + 2\mu/\beta a)/(1 + 3\mu/\beta a).$$

If there is no slipping the value of V is

$$V = \frac{2ga^2}{9\mu}(\sigma - \rho) \quad\text{.......................(45)}.$$

The preceding formula has been applied by Prof. Stokes to show that the viscosity of the air is sufficient to account for the suspension of the clouds.

496. We shall now determine the steady motion of liquid which surrounds a sphere, which is constrained to rotate with uniform angular velocity about a fixed diameter.

By (30) the equation for W is

$$\frac{d^2W}{dr^2}+\frac{2}{r}\frac{dW}{dr}-\frac{2W}{r^2}=0,$$

the solution of which is

$$W=Ar+B/r^2.$$

Since W must not be infinite when $r=\infty$, $A=0$, whence

$$w=Br^{-2}\sin\theta.$$

The surface condition (28) gives

$$B=\omega a^3\left(1+\frac{3\mu}{\beta a}\right)^{-1}.$$

Whence

$$w=\frac{\omega a^3}{r^2}\sin\theta\left(1+\frac{3\mu}{\beta a}\right)^{-1} \quad\text{.................(46).}$$

If $\beta=\infty$ this becomes

$$w=\frac{\omega a^3}{r^2}\sin\theta \quad\text{.......................(47).}$$

which is Prof. Stokes' result.

The couple which must be applied to the sphere in order to maintain the motion is

$$G=-2\pi\mu a^3\frac{d}{dr}\left(\frac{w}{r}\right)\int_0^\pi \sin^3\theta d\theta,$$

$$=8\pi\mu\omega a^2(1+3\mu/\beta a)^{-1}.$$

In obtaining the preceding result we have tacitly assumed that the stream lines are concentric circles, whose centres lie on the axis of rotation. Prof. Stokes has however pointed out,—"that permanent motion in annuli is impossible, whatever may be the law of friction between the sphere of the liquid, and it is therefore necessary to suppose that the particles move in planes passing through the axis of rotation, while at the same time they move around it. In fact it is easy to see that from the excess of centrifugal force in the neighbourhood of the equator of the revolving sphere, the particles in that part will recede from the sphere, and approach it again at the poles, and this circulating motion will be combined with a motion about the axis. If however we leave the centrifugal force out of consideration, the motion in annuli becomes possible[1]."

[1] *Math. and Phys. Papers*, vol. I. p. 103.

Steady Motion of an Ellipsoid.

497. By means of equation (42) it can easily be shown that if the axis of x be the direction of motion of the sphere, the component velocities u, v, w parallel to the axes of x, y, z are determined by the equations

$$\begin{aligned} u &= \tfrac{1}{4}V\left(\frac{3a}{r}+\frac{a^3}{r^3}\right)+\tfrac{3}{4}V\left(1-\frac{a^2}{r^2}\right)\frac{ax^2}{r^3}, \\ v &= \tfrac{3}{4}V\left(1-\frac{a^2}{r^2}\right)\frac{axy}{r^3}, \\ w &= \tfrac{3}{4}V\left(1-\frac{a^2}{r^2}\right)\frac{axz}{r^3}. \end{aligned}$$

The preceding formulæ suggested to Oberbeck[1] the corresponding results in the case of an ellipsoid, which moves parallel to one of its principal axes.

Let Ω be the potential of an ellipsoid of unit density, so that with the notation of § 147,

$$\Omega = \tfrac{1}{2}\left(A_\lambda x^2 + B_\lambda y^2 + C_\lambda z^2\right) - H_\lambda ;$$

and let

$$\left.\begin{aligned} u &= \alpha\left(x\frac{dH_\lambda}{dx} - H_\lambda + \beta\frac{d^2\Omega}{dx^2}\right) \\ v &= \alpha\left(x\frac{dH_\lambda}{dy} + \beta\frac{d^2\Omega}{dxdy}\right) \\ w &= \alpha\left(x\frac{dH_\lambda}{dz} + \beta\frac{d^2\Omega}{dxdz}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(48),$$

where α, β are constants. It can easily be shown that these values of u, v, w satisfy the equation of continuity, and vanish at infinity.

If the ellipsoid move parallel to x with velocity V, and there is no slipping, the surface conditions are

$$u = V, \quad v = 0, \quad w = 0.$$

If p be the perpendicular from the centre on to the tangent plane at (x, y, z), and the unsuffixed letters denote the surface values of A, B, C, H,

$$\frac{dH}{dx} = -\frac{2\pi p^2 x}{a^2}, \quad \frac{d^2\Omega}{dx^2} = A - \frac{4\pi p^2 x^2}{a^4},$$

[1] *Borch.* vol. LXXXI. p. 62.

whence $$V = \alpha\left\{A\beta - H - \frac{2\pi p^2 x^2}{a^2}\left(1 + \frac{2\beta}{a^2}\right)\right\},$$

and therefore

$$\beta = -\tfrac{1}{2}a^2, \quad \alpha = -\frac{2V}{Aa^2 + 2H} \quad \text{..........} \quad (49);$$

also the preceding values of α and β make v and w each zero at the surface.

If X be the force required to maintain the motion,

$$X = \iint (Pl + Um + Tn)\, dS.$$

Now let us suppose that the liquid is bounded by a very large sphere whose radius r is ultimately made infinite; then by § 482

$$X = -\iint (P'l' + U'm' + T'n')\, dS,$$

where the accents refer to the spherical boundary.

At a great distance from the origin $H_\lambda = E/r$, where E is the charge due to a distribution of electricity upon the ellipsoid of density $\frac{1}{2}p$; also the coefficients of β in (48) are of the order r^{-3} and therefore ultimately vanish. We thus obtain

$$u = -E\alpha\left(\frac{1}{r} + \frac{x^2}{r^3}\right), \qquad v = -\frac{E\alpha xy}{r^3}, \qquad w = -\frac{E\alpha xz}{r^3},$$

whence $$p = -2\mu E\alpha x/r^3,$$

and therefore

$$-P' = -p + 2\mu\frac{du}{dx} = 6E\alpha\mu x^3/r^5,$$

$$-U' = 6E\alpha\mu x^2 y/r^5,$$

$$-T' = 6E\alpha\mu x^2 z/r^5,$$

therefore $$P'l' + U'm' + T'n' = 6E\alpha\mu x^2/r^4.$$

Hence $$X = -12\pi E\alpha\mu\int_0^\pi \cos^2\theta \sin\theta d\theta,$$

$$= -4\pi E\alpha\mu,$$

$$= \frac{8\pi\mu E V}{Aa^2 + 2H}.$$

Motion of a Cylinder[1].

498. We must in the next place consider the small oscillations and steady motion of a circular cylinder; and we shall commence with the case of a cylindrical pendulum, which is performing small oscillations along a straight line.

Let u, v be the velocities of the liquid parallel to fixed rectangular axes; the equations of motion are

$$\left.\begin{aligned} \frac{du}{dt} &= \frac{dQ}{dx} + \nu\nabla^2 u \\ \frac{dv}{dt} &= \frac{dQ}{dy} + \nu\nabla^2 v \end{aligned}\right\} \quad \text{......................(50)},$$

where $Q = -p/\rho - V$; the squares and products of the velocities being neglected.

Also if ψ be the current function

$$u = d\psi/dy, \quad v = -d\psi/dx;$$

whence eliminating Q we obtain

$$\nabla^2\left(\nabla^2 - \frac{1}{\nu}\frac{d}{dt}\right)\psi = 0 \quad \text{....................(51)}.$$

This equation will be satisfied by putting

$$\psi = \psi_1 + \psi_2,$$

where

$$\nabla^2\psi_1 = 0 \quad \text{.........................(52)},$$

$$\left(\nabla^2 - \frac{1}{\nu}\frac{d}{dt}\right)\psi_2 = 0 \quad \text{.....................(53)}.$$

Substituting for u, v in terms of ψ in (50) we obtain,

$$-dQ = \nu dx\,\frac{d}{dy}\left(\nabla^2 - \frac{1}{\nu}\frac{d}{dt}\right)\psi - \nu dy\,\frac{d}{dx}\left(\nabla^2 - \frac{1}{\nu}\frac{d}{dt}\right)\psi,$$

which becomes by (52) and (53),

$$-dQ = \left(\frac{d^2\psi_1}{dxdt}\,dy - \frac{d^2\psi_1}{dydt}\,dx\right) \quad \text{................(54)}.$$

Let R, Θ be the velocities of the liquid along and perpendicular

[1] Stokes, *Trans. Camb. Phil. Soc.* vol. IX.

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to the radius vector, a the radius of the cylinder; then changing to polar coordinates we have

$$R = \frac{1}{r}\frac{d\psi}{d\theta}, \quad \Theta = -\frac{d\psi}{dr},$$

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{1}{r^2}\frac{d^2}{d\theta^2},$$

whence (54) becomes

$$-dQ = \frac{d}{dt}\left(\frac{d\psi_1}{dr}\, r d\theta - \frac{d\psi_1}{r d\theta}\, dr\right) \quad \text{.............(55).}$$

If V be the velocity of the cylinder, the surface conditions are

$$\frac{d\psi}{d\theta} = Va\cos\theta, \quad \frac{d\psi}{dr} = V\sin\theta \quad \text{..............(56),}$$

when $r = a$. Equations (56) show that θ must enter into ψ in the form of the factor $\sin\theta$; also if we assume that the time factor is $\epsilon^{\lambda^2\nu t}$, we may put

$$\psi_1 = \epsilon^{\lambda^2\nu t}\sin\theta\chi_1(r), \quad \psi_2 = \epsilon^{\lambda^2\nu t}\sin\theta\chi_2(r), \quad V = c\epsilon^{\lambda^2\nu t} \text{...(56 A).}$$

Substituting in (52), (53) and (56) we obtain

$$\chi_1'' + \chi_1'/r - \chi_1/r^2 = 0 \quad \text{...................(57),}$$

$$\chi_2'' + \chi_2'/r - \chi_2/r^2 - \lambda^2\chi_2 = 0 \quad \text{.............. (58),}$$

$$\chi_1(a) + \chi_2(a) = ac, \quad \chi_1'(a) + \chi_2'(a) = c \quad \text{........(59).}$$

The integral of (57) is

$$\chi_1 = A/r + Br \quad \text{......................(60),}$$

whence since $\chi_1 = 0$ when $r = \infty$, $B = 0$.

·499 Since χ_2 must vanish when $r = \infty$, the proper solution of (58) is $\chi_2 = K_1(\lambda r)$ where K_1 is a Bessel's function of the second kind of order unity; but since λ is a complex quantity, the definite integral form of K_1 is not a convenient expression, and we shall therefore proceed to find one suited to our purpose.

Let $\chi_2 = du/dr$; substituting in (58) and integrating we obtain

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \lambda^2 u = 0 \quad \text{...................(61).}$$

If the equation

$$\frac{d^2u'}{dr^2} + \frac{1}{r}\frac{du'}{dr} - \left(\lambda^2 + \frac{n^2}{r^2}\right)u' = 0 \quad \text{.............(62),}$$

be integrated by series, the result is

$$u' = Ar^n \left\{1 + \frac{\lambda^2 r^2}{2.(2+2n)} + \frac{\lambda^4 r^4}{2.4.(2+2n).(4+2n)} + \ldots\ldots\right\}$$
$$+ \frac{B}{r^n}\left\{1 + \frac{\lambda^2 r^2}{2.(2-2n)} + \frac{\lambda^4 r^4}{2.4.(2-2n).(4-2n)} + \ldots\ldots\right\}.$$

The latter series fails when n is an integer since it becomes infinite; and when n is zero the two series become identical. Let us therefore denote the first series by $f(n)$ and the second by $f(-n)$, then by Maclaurin's theorem

$$u' = Af(n) + Bf(-n)$$
$$= (A+B)f(0) + (A-B)\,nf'(0) + (A+B)\frac{n^2}{2\,!}f''(0) + \ldots\ldots,$$

whence choosing new arbitrary constants, the value of u' when $n = 0$ will be

$$u' = u = Cf(0) + Df'(0).$$

Now

$$f'(n) = r^n \log r \left\{1 + \frac{\lambda^2 r^2}{2.(2+2n)} + \frac{\lambda^4 r^4}{2.4.(2+2n).(4+2n)} + \ldots\ldots\right\}$$
$$+ r^n \frac{d}{dn}\left\{1 + \frac{\lambda^2 r^2}{2(2+2n)} + \frac{\lambda^4 r^4}{2.4.(2+2n).(4+2n)} + \ldots\ldots\right\}.$$

If v denote the coefficient of $(\lambda r)^{2m}$ in the above series

$$\frac{1}{v}\frac{dv}{dn} = \frac{d(\log v)}{dn} = -\frac{1}{1+n} - \frac{1}{2+n} - \ldots - \frac{1}{m+n};$$

whence $$\left(\frac{dv}{dn}\right)_0 = -v_0 S_m,$$

where $$S_m = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{m};$$

and therefore

$$f'(0) = \log r \left\{1 + \frac{\lambda^2 r^2}{2^2} + \frac{\lambda^4 r^4}{2^2.4^2} + \frac{\lambda^6 r^6}{2^2.4^2.6^2} + \ldots\ldots\right\}$$
$$- \left\{\frac{\lambda^2 r^2}{2^2} + \frac{\lambda^4 r^4}{2^2.4^2} S_2 + \frac{\lambda^6 r^6}{2^2.4^2.6^2} S_3 + \ldots\ldots\right\};$$

whence the solution of (61) is

$$u = (C + D\log r)\left(1 + \frac{\lambda^2 r^2}{2^2} + \frac{\lambda^4 r^4}{2^2.4^2} + \ldots\ldots\right)$$
$$- D\left(\frac{\lambda^2 r^2}{2^2} + \frac{\lambda^4 r^4}{2^2.4^2} S_2 + \ldots\ldots\right) \ldots\ldots\ldots\ldots\ldots(63).$$

We may also integrate (61) in a series of descending powers of r by assuming

$$u = \epsilon^{\pm\lambda r}(Ar^\alpha + Br^\beta + \ldots\ldots),$$

and we shall obtain

$$u = C'\epsilon^{-\lambda r}r^{-\frac{1}{2}}\left\{1 - \frac{1^2}{2.4\lambda r} + \frac{1^2.3^2}{2.4.(4\lambda r)^2} - \frac{1^2.3^2.5^2}{2.4.6(4\lambda r)^3} + \ldots\ldots\right\}$$

$$+ D'\epsilon^{\lambda r}r^{\frac{1}{2}}\left\{1 + \frac{1}{2.4\lambda r} + \frac{1^2.3^2}{2.4.(4\lambda r)^2} + \frac{1^2.3^2.5^2}{2.4.6(4\lambda r)^3} + \ldots\ldots\right\} \ldots(64).$$

A third form of the integral of (61) may be obtained as follows. One of the solutions is

$$I_n(\lambda r) = r^n \int_0^{\frac{1}{2}\pi} \cosh(\lambda r\cos\phi)\sin^{2n}\phi\, d\phi = f(n),$$

whence $$f'(0) = \int_0^{\frac{1}{2}\pi} \cosh(\lambda r\cos\phi)\log(r\sin^2\phi)\,d\phi,$$

and therefore

$$u = \int_0^{\frac{\pi}{2}} \{C'' + D''\log(r\sin^2\phi)\}\cosh(\lambda r\cos\phi)\,d\phi \ldots (65).$$

Also comparing (63) and (65) we shall obtain

$$C + D\log r = \tfrac{1}{2}\pi(C'' + D''\log r) + \pi D''\log\tfrac{1}{2},$$

whence $$C = \tfrac{1}{2}\pi C'' + \pi D''\log\tfrac{1}{2},\quad D = \tfrac{1}{2}\pi D'' \ldots\ldots\ldots\ldots(66).$$

We must now find the relation between C'' and D'' in order that $u = 0$ when $r = \infty$.

When r is very large the limit of $\epsilon^{-r}\log r$ is zero, also since $\sin\phi$ and $\cos\phi$ can never be greater than unity throughout the range of integration, ϕ will be very small compared with r. We may therefore replace the limits $\frac{1}{2}\pi$ and 0 in the integral (65) by ω and 0, where ω is a very small positive quantity which ultimately vanishes when $r = \infty$.

Let $\cos\phi = 1 - x$, so that

$$\sin^2\phi = 2x(1 - \tfrac{1}{2}x),\quad d\phi = (2x - x^2)^{-\frac{1}{2}}dx = (1 - \tfrac{1}{4}x + \ldots\ldots)(2x)^{-\frac{1}{2}}dx,$$

then the limits of x will be x_1 and 0 where $x_1 = 1 - \cos\omega$. Whence the limit of the integral (65) when r is large is

$$\epsilon^{\lambda r}\int_0^{x_1}(C'' + D''\log 2rx)\,\epsilon^{-\lambda rx}(2x)^{-\frac{1}{2}}dx.$$

Let $x = y/r$, then the limits of y are rx_1 and 0, the former of which becomes infinite with r, whence

$$\text{limit of integral} = (2r)^{-\frac{1}{2}}\epsilon^{\lambda r}\int_0^\infty (C'' + D''\log 2y)\,\epsilon^{-\lambda y}\,y^{-\frac{1}{2}}\,dy.$$

Now $\displaystyle\int_0^\infty \frac{\epsilon^{-x}\,dx}{\sqrt{x}} = \sqrt{\pi}$, and if we differentiate both sides of the equation

$$\int_0^\infty \epsilon^{-x}x^{s-1}\,dx = \Gamma(s),$$

with respect to s and then put $s = \frac{1}{2}$, we obtain

$$\int_0^\infty \epsilon^{-x}x^{-\frac{1}{2}}\log x dx = \Gamma'(\tfrac{1}{2}),$$

putting $x = \lambda y$, the required limit becomes

$$u = (\pi/2\lambda r)^{\frac{1}{2}}\epsilon^{\lambda r}\left[C'' + D''\left\{\pi^{-\frac{1}{2}}\Gamma'(\tfrac{1}{2}) - \log\tfrac{1}{2}\lambda\right\}\right].$$

Comparing this with (64) we obtain

$$D' = (\pi/2\lambda)^{\frac{1}{2}}\left\{C'' + D''\left[\pi^{-\frac{1}{2}}\Gamma'(\tfrac{1}{2}) - \log\tfrac{1}{2}\lambda\right]\right\} \quad\text{......(67).}$$

Now in order that u may vanish when $r = \infty$ we must have $D' = 0$, whence the required relation between C'' and D'' is

$$C'' = D''\left\{\log\tfrac{1}{2}\lambda - \pi^{-\frac{1}{2}}\Gamma'(\tfrac{1}{2})\right\},$$

and therefore by (66) the required relation between C and D is

$$C = D\left\{\log\tfrac{1}{8}\lambda - \pi^{-\frac{1}{2}}\Gamma'(\tfrac{1}{2})\right\} \quad\text{.................(68).}$$

Putting $u = F(r)$ we obtain from (59)

$$A/a + F'(a) = ac, \quad -A/a + aF''(a) = ac \quad\text{...... (69),}$$

whence

$$\frac{a^2c + A}{a^2c - A} = \frac{aF''(a)}{F'(a)} \quad\text{..................... (70),}$$

(63), (68) and (69) completely determine A, C and D.

500. Let us now suppose that the cylinder is a pendulum oscillating under the action of gravity; and let Z be the resistance experienced by it per unit of length, then

$$Z = a\int_0^{2\pi} (-P\cos\theta + U\sin\theta)_a\,d\theta,$$

where P and U are given by Example 5, Ch. XX. Adding (52) and (53) we obtain

$$\frac{d^2\psi}{dr^2} = -\frac{1}{r}\frac{d\psi}{dr} - \frac{1}{r}\frac{d^2\psi}{d\theta^2} + \frac{1}{\nu}\frac{d\psi_2}{dt}.$$

Using this equation together with (56) and (59) we obtain

$$\left(\frac{dR}{dr}\right)_a = -\frac{1}{a^2}\left(\frac{d\psi}{d\theta}\right)_a + \frac{1}{a}\left(\frac{d^2\psi}{drd\theta}\right)_a = 0,$$

$$\frac{1}{a}\left(\frac{dR}{d\theta}\right)_a = \frac{1}{a^2}\left(\frac{d^2\psi}{d\theta^2}\right)_a = -\frac{V\sin\theta}{a} = \frac{\Theta}{a},$$

$$\left(\frac{d\Theta}{dr}\right)_a = -\left(\frac{d^2\psi}{dr^2}\right)_a = \frac{1}{a}\left(\frac{d\psi}{dr}\right)_a + \frac{1}{a^2}\left(\frac{d^2\psi}{d\theta^2}\right)_a - \frac{1}{\nu}\left(\frac{d\psi_2}{dt}\right)_a = -\frac{1}{\nu}\left(\frac{d\psi_2}{dt}\right)_a,$$

whence $$Z = a\int_0^{\pi}\left\{p\cos\theta - \rho\frac{d\psi_2}{dt}\sin\theta\right\}_a d\theta.$$

By (55) $$\frac{1}{\rho}\frac{dp}{d\theta} + ga\cos\theta = -\frac{dQ}{d\theta} = a\frac{d^2\psi_1}{drdt}.$$

Therefore integrating the first term by parts, we obtain

$$\int_0^{2\pi} p\cos\theta\, d\theta = -\int_0^{2\pi}\sin\theta\frac{dp}{d\theta}d\theta$$

$$= -a\rho\int_0^{2\pi}\sin\theta\frac{d^2\psi_1}{drdt}d\theta,$$

whence $$Z = -\rho a\int_0^{2\pi}\frac{d}{dt}\left\{a\frac{d\psi_1}{dr} + \psi_2\right\}_a \sin\theta\, d\theta.$$

Putting $\lambda^2\nu = \iota n$ and substituting the values of ψ_1 and ψ_2 from (56 A.), we obtain

$$Z = \pi\rho a n\iota\,\{A/a - F'(a)\}\,\epsilon^{\iota nt}.$$

By (69) $F'(a) = ac - A/a$ and by (70)

$$A = \frac{aF''(a) - F'(a)}{aF''(a) + F'(a)}a^2c.$$

Therefore

$$Z = -M'cn\iota\left\{1 - 2\frac{aF''(a) - F'(a)}{aF''(a) + F'(a)}\right\}\epsilon^{\iota nt},$$

where M' is the mass of the liquid displaced. Since F satisfies the differential equation (61),

$$Z = M'cn\iota\left\{1 - \frac{4F'(a)}{\lambda^2 aF(a)}\right\}\epsilon^{\iota nt}$$

$$= M'cn\iota\,\epsilon^{\iota nt}(K - \iota K') \quad \text{(say)},$$

where K and K' are real. Whence if l be the length of the pendulum, the equation of motion of the cylinder is

$$Ml\ddot{\xi} + M'K\ddot{\xi} + M'K'n\dot{\xi} + (M - M')g\xi = 0.$$

The second term of this equation is the part of Z which alters the time of oscillation, and the third term is the part which diminishes the arc of oscillation.

For the calculation of the quantities K, K' we must refer the reader to Professor Stokes' memoir.

501. We shall now show that when a cylinder is moving in a straight line, steady motion is impossible.

Putting $\psi = \psi' \sin\theta$ in (51), it follows that the equation for determining the value of ψ' in steady motion is

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\right)^2 \psi' = 0 \ldots\ldots\ldots\ldots\ldots\ldots(71),$$

whence
$$\frac{d^2\psi'}{dr} + \frac{1}{r}\frac{d\psi'}{dr} - \frac{\psi'}{r^2} = Ar + B/r,$$

the integral of which may be written

$$\psi' = Ar\log r + Cr + Dr^{-1} + Br^3.$$

We therefore obtain

$$\left.\begin{aligned} R &= \frac{1}{r}\frac{d\psi}{d\theta} = (A\log r + Br^2 + C + Dr^{-2})\cos\theta \\ \Theta &= -\frac{d\psi}{dr} = (Dr^{-2} - C - 3Br^2 - A\log r - A)\sin\theta \end{aligned}\right\} \ldots(72).$$

Let us now suppose that the motion is reversed, so that the cylinder is at rest, whilst the liquid is streaming past it, the velocity of the latter at infinity being equal to $-V$. The equations of condition are

$$R = 0, \qquad \Theta = 0, \qquad \text{when } r = a \ldots\ldots (73),$$

$$R = -V\cos\theta, \quad \Theta = V\sin\theta, \quad \text{when } r = \infty \ldots\ldots(74).$$

The first of equations (74) requires that $A = 0$, $B = 0$, $C = -V$, which also satisfies the second, and we are thus left with one disposable constant to satisfy equations (73); and since both these equations cannot be satisfied by the same value of D, steady motion is impossible.

502. When the motion of a liquid is symmetrical about a point and is in two dimensions, it follows from the second of equations (23) of § 471 that the velocity is determined by the equation

$$\frac{dv}{dt} = \nu\left(\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2}\right) \text{..................} (75).$$

If therefore the motion is steady, the value of v is

$$v = Ar + B/r.$$

Hence if a cylinder is surrounded with viscous liquid and made to rotate with angular velocity ω, the value of v after the motion has become steady is

$$v = a^2\omega/r.$$

If on the other hand the cylinder is filled with liquid, the value of v is ωr, and therefore the liquid rotates like a rigid body.

EXAMPLES.

1. When a sphere is moving with uniform velocity along a straight line, prove that after the motion has become steady, the vorticity at any point of a vortex line is inversely proportional to the cube of the distance of that point from the centre of the sphere.

2. A doublet of strength m is situated at the centre of a sphere of radius a. Prove that after the motion has become steady, the radial and transversal velocities of the liquid are respectively equal to

$$m\left(\frac{2}{r^3} - \frac{5}{a^3} + \frac{3r^2}{a^5}\right)\cos\theta, \text{ and } m\left(\frac{1}{r^3} + \frac{5}{a^3} - \frac{6r^2}{a^5}\right)\sin\theta.$$

3. The space between two concentric spheres is filled with viscous liquid, and the spheres are made to rotate with different angular velocities about the same diameter. Assuming that the particles of liquid move in planes perpendicular to the axis of rotation, and that there is no slipping, find the velocity of the liquid after the motion has become steady; and prove that if the

inner sphere is at rest, the couple which must be applied to the outer one in order to maintain the motion is equal to

$$\frac{8\pi\mu\omega a^2 b^3}{a^3 - b^3},$$

where a and b are the radii of the outer and inner spheres, and ω the angular velocity of the former.

4. The space between two concentric cylinders of radii a and b, is filled with viscous liquid, and the cylinders are constrained to rotate with angular velocities ω_1, ω_2; prove that if ω be the angular velocity of a liquid particle at a distance r from the axis after the motion has become steady

$$\frac{\omega_1 - \omega}{\omega_1 - \omega_2} = \frac{1 - (a/r)^2}{1 - (a/b)^2}.$$

5. A cylinder of length l and radius a, which is surrounded by viscous liquid, is made to rotate with uniform angular velocity ω. Prove that if slipping takes place, the couple which must be applied to the cylinder to maintain the motion when steady, is equal to

$$\frac{4\pi\mu\omega a^2 l}{1 + 2\mu/\beta a}.$$

6. A cylinder of radius a is filled with viscous liquid and constrained to rotate so that the angular velocity at any time is $\omega \sin mt$; prove that if there is no slipping at the surface, the current function is

$$\psi = \tfrac{1}{2} a\omega \,(P \cos mt + Q \sin mt),$$

where $$P + \iota Q = (1 - \iota) k \frac{J_0 \{k (1 - \iota) r\}}{J_0 \{k (1 - \iota) a\}}, \quad \text{and} \quad k^2 = m/2\nu.$$

7. A long right circular cylinder is rotating with uniform angular velocity ω inside a concentric cylinder which is at rest, the space between the cylinders being filled with viscous liquid; show that the couple on the cylinder per unit of length is

$$\frac{4\mu\omega a^2 b^2}{a^2 - b^2},$$

where a and b are the radii of the outer and inner cylinders.

8. The inner of two confocal ellipsoids of revolution, the space between which is filled with viscous liquid, is made to rotate with angular velocity ω about its axis, the outer one being at rest; prove that the velocities of the liquid are

$$u = -\omega y (A - A_1)/(A_2 - A_1), \quad v = \omega x (A - A_1)/(A_2 - A_1),$$

where
$$A = \int_0^{\mu} \frac{d\psi}{(a^2 + \psi)^2 (c^2 + \psi)^{\frac{1}{2}}},$$

and A_1, A_2 are the values of A at the outer and inner ellipsoid respectively.

9. A thin circular disc is oscillating in a viscous liquid by means of a torsion fibre. Prove that the equation of motion of the disc is

$$(I + \pi\rho a^4 k)\ddot{\theta} + \pi\rho a^4 k\dot{\theta} + n_1^2 I\theta = 0,$$

where I is the moment of inertia of the disc, a its radius, ρ the density of the liquid, $k^2 = n/2\nu$, and n_1 is what n would become if the liquid were absent.

Integrate this equation, and explain how the result may be used to determine the coefficient of viscosity.

CHAPTER XXII.

ON THE MOTION OF A SPHERE IN A VISCOUS LIQUID.

503. In the preceding Chapter we considered the steady motion and small oscillations of a sphere and a cylinder in a viscous liquid; we shall now proceed to investigate the motion of a sphere which is surrounded by a viscous liquid of unlimited extent, and which is moving in a straight line under the action of a constant force such as gravity[1].

The mathematical difficulties of integrating the general equations of motion when the terms involving the squares and products of the velocities are retained, will compel us to omit them throughout the whole of this Chapter. This is no doubt legitimate provided we confine our attention to the consideration of slow motions; but when the motion is not slow it must be confessed that the assumption that these terms can be neglected is of a questionable character. It will be seen that the results which we shall obtain give a better representation of the motion which actually takes place, than those which are obtained from the ordinary theory of a frictionless liquid; and it should also be noticed that when the liquid is frictionless the terms involving the squares and products of the velocities do not contribute anything to the resistance experienced by the sphere; and it is therefore not impossible that when the viscosity is small, the effect of these terms may be unimportant compared with those retained. Since the equations of motion can be reduced to a comparatively simple form when these terms are omitted, I am inclined to think that the procedure which would be most likely to be successful in

[1] *Phil. Trans.* 1888, p. 43.

advancing our theoretical knowledge of the motion of solid bodies in viscous liquids, would be to neglect these terms in the first instance, in the hope that the imperfect solutions which are thereby obtained, may hereafter suggest a more satisfactory method of dealing with such problems.

Motion of a Sphere under the Action of Gravity.

504. Let us suppose that a sphere of radius a, is surrounded by a viscous liquid which is initially at rest, and let the sphere be constrained to move with uniform velocity V, in a straight line. If the squares and products of the velocity of the liquid are neglected, we have shown in the previous Chapter that the current function ψ must satisfy the differential equation

$$D\left(D - \frac{1}{\nu}\frac{d}{dt}\right)\psi = 0 \quad\text{....................(1)},$$

where $$D = \frac{d^2}{dr^2} + \frac{\sin\theta}{r^2}\frac{d}{d\theta}\left(\operatorname{cosec}\theta\,\frac{d}{d\theta}\right)$$

and (r, θ) are polar coordinates of a point referred to the centre of the sphere as origin.

Let R, Θ be the component velocities of the liquid along and perpendicular to the radius vector; then, if we assume that no slipping takes place at the surface of the sphere, the surface conditions are

$$R = \frac{1}{a^2\sin\theta}\frac{d\psi}{d\theta} = V\cos\theta \quad\text{....................(2)},$$

$$\Theta = -\frac{1}{a\sin\theta}\frac{d\psi}{dr} = -V\sin\theta \quad\text{................(3)}.$$

Also, at infinity R and Θ must both vanish.

These equations can be satisfied by putting

$$\psi = (\psi_1 + \psi_2)\sin^2\theta \quad\text{.......................(4)},$$

where ψ_1 and ψ_2 are functions of r and t, which respectively satisfy the equations

$$\frac{d^2\psi_1}{dr^2} - \frac{2\psi_1}{r^2} = 0 \quad\text{.......................(5)},$$

$$\frac{d^2\psi_2}{dr^2} - \frac{2\psi_2}{r^2} = \frac{1}{\nu}\frac{d\psi_2}{dt} \quad\text{....................(6)}.$$

The proper solution of (5) is $\psi_1 = f(t)/r$, which it will be convenient to write in the form

$$\psi_1 = \frac{\sqrt{\pi}}{2r\sqrt{(\nu t)}}\int_0^\infty \chi(\alpha)\exp(-\alpha^2/4\nu t)\,d\alpha \quad\text{.........} (7),$$

where $\chi(\alpha)$ is an arbitrary function, which will hereafter be determined.

In order to obtain the solution of (6), let us put $\psi_2 = r\epsilon^{-\lambda^2\nu t}\,dw/dr$, where w is a function of r alone; substituting in (6), and integrating, we obtain

$$rw = A\cos\lambda(r-a+\alpha),$$

where a is the radius of the sphere and A and α are the constants of integration. Whence a particular solution of (6) is

$$\psi_2 = Ar\frac{d}{dr}\frac{\epsilon^{-\lambda^2\nu t}}{r}\cos\lambda(r-a+\alpha).$$

Integrating this with respect to λ between the limits ∞ and 0, and then changing A into $F(\alpha)$ and integrating the result with respect to α between the same limits, we obtain

$$\psi_2 = \frac{r\sqrt{\pi}}{2\sqrt{(\nu t)}}\frac{d}{dr}\int_0^\infty \frac{F(\alpha)}{r}\exp\left\{-\frac{(r-a+\alpha)^2}{4\nu t}\right\}d\alpha.$$

Performing the differentiation and then integrating by parts, we obtain

$$\psi_2 = -\tfrac{1}{2}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty\left\{\frac{F(\alpha)}{r}+F'(\alpha)\right\}\exp\left\{-\frac{(r-a+\alpha)^2}{4\nu t}\right\}d\alpha$$
$$+\tfrac{1}{2}\sqrt{\frac{\pi}{\nu t}}\left[F(\alpha)\exp\left\{-\frac{(r-a+\alpha)^2}{4\nu t}\right\}\right]_0^\infty.$$

We shall presently show that it is possible to determine $F(\alpha)$, so that $F(0)=0$, and $F(\alpha)\,\epsilon^{-\alpha^2}=0$ when $\alpha=\infty$; hence the term in square brackets will vanish at both limits, and we obtain

$$\psi = \frac{\sin^2\theta}{2r}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty \chi(\alpha)\exp\left(-\frac{\alpha^2}{4\nu t}\right)d\alpha$$
$$-\tfrac{1}{2}\sin^2\theta\sqrt{\frac{\pi}{\nu t}}\int_0^\infty\left\{\frac{F(\alpha)}{r}+F'(\alpha)\right\}\exp\left\{-\frac{(r-a+\alpha)^2}{4\nu t}\right\}d\alpha \text{...} (8).$$

We must now determine the functions χ and F so as to satisfy the surface conditions (2) and (3).

Equation (2) will be satisfied if

$$\chi(\alpha) - F(\alpha) - aF'(\alpha) = V\alpha^3/\pi \quad\text{..............} (9).$$

Equation (3) requires that

$$Va^3 = -\tfrac{1}{2}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty \chi(\alpha)\exp\left(-\frac{\alpha^2}{4\nu t}\right)d\alpha$$
$$+\tfrac{1}{2}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty F(\alpha)\exp\left(-\frac{\alpha^2}{4\nu t}\right)d\alpha$$
$$-\tfrac{1}{2}a\sqrt{\frac{\pi}{\nu t}}\int_0^\infty \{F(\alpha)+aF'(\alpha)\}\frac{d}{d\alpha}\exp\left(-\frac{\alpha^2}{4\nu t}\right)d\alpha.$$

Integrating the last term by parts, the preceding equation becomes

$$Va^3 = \tfrac{1}{2}(\pi/\nu t)^{\frac{1}{2}}\int_0^\infty \{-\chi(\alpha)+F(\alpha)+aF'(\alpha)+a^2F''(\alpha)\}$$
$$\times \exp(-\alpha^2/4\nu t)\,d\alpha \quad\text{.................(10)},$$

provided, $\{F(\alpha)+aF'(\alpha)\}\exp(-\alpha^2/4\nu t)$ vanishes at both limits. This requires that $F(0)=F'(0)=0$, and that $F(\alpha)\epsilon^{-\alpha^2}$ and $F'(\alpha)\epsilon^{-\alpha^2}$ should each vanish when $\alpha=\infty$. When this is the case (10) will be satisfied if

$$-\chi(\alpha)+F(\alpha)+aF'(\alpha)+a^2F''(\alpha)=2Va^3/\pi \quad\text{......(11)}.$$

Whence by (9) $\quad F''(\alpha)=3Va/\pi$

and, therefore, $\quad F(\alpha)=3Va\alpha^2/2\pi+C\alpha+D.$

The conditions that $F(0)=F'(0)=0$ require that $C=D=0$; whence

$$F(\alpha)=3Va\alpha^2/2\pi,\quad \chi(\alpha)=Va\pi^{-1}(\tfrac{3}{2}\alpha^2+3a\alpha+a^2).$$

Also the preceding value of $F(\alpha)$ satisfies the conditions that $F(\alpha)\epsilon^{-\alpha^2}$, and $F'(\alpha)\epsilon^{-\alpha^2}$ should each vanish when $\alpha=\infty$; whence all the conditions are satisfied, and we finally obtain

$$\psi=\frac{Va\sin^2\theta}{2r\sqrt{(\pi\nu t)}}\int_0^\infty(\tfrac{3}{2}\alpha^2+3a\alpha+\tfrac{1}{2}a^2)\exp\left(-\frac{\alpha^2}{4\nu t}\right)d\alpha$$
$$-\frac{3Va\sin^2\theta}{2\sqrt{(\pi\nu t)}}\int_0^\infty\left(\frac{\alpha^2}{2r}+\alpha\right)\exp\left\{-\frac{(r-a+\alpha)^2}{4\nu t}\right\}d\alpha\quad\text{...(12)}.$$

The first integral can be evaluated; in the second put $r-a+\alpha=2u\sqrt{(\nu t)}$ and we obtain

$$\psi=\frac{Va\sin^2\theta}{2r}\{3\nu t+6a\sqrt{(\nu t/\pi)}+a^2\}$$
$$-\frac{3Va\sin^2\theta}{\sqrt{\pi}}\int_{\frac{r-a}{2\sqrt{(\nu t)}}}^{\infty}\left\{\frac{1}{2r}\{2u\sqrt{(\nu t)}-r+a\}^2+2u\sqrt{(\nu t)}-r+a\right\}\epsilon^{-u^2}du$$
$$\text{............(13)}.$$

505. When $t=0$ the second integral vanishes, whence the initial value of ψ is

$$\psi = \frac{Va^3 \sin^2\theta}{2r},$$

which is the known value of ψ in the case of a frictionless liquid, as ought to be the case.

When t is very large, we may put $t=\infty$ in the lower limit of the second integral, which then

$$= -\frac{3Va\sin^2\theta}{r\sqrt{\pi}} \int_0^\infty \{2u^2\nu t + 2au\sqrt{(\nu t)} + \tfrac{1}{2}(a^2 - r^2)\} \epsilon^{-u^2}\, du,$$

$$= -\frac{Va\sin^2\theta}{2r}\{3\nu t + 6a\sqrt{(\nu t)} + \tfrac{3}{2}(a^2-r^2)\},$$

whence $$\psi = \tfrac{1}{4} Va^2 \sin^2\theta \left(\frac{3r}{a} - \frac{a}{r}\right).$$

This equation gives the value of ψ after a sufficient time has elapsed for the motion to have become steady, and agrees with the result obtained in § 494.

506. Let v_t be any solution of the partial differential equation

$$\phi\left(\frac{d}{dr}\right) u = \frac{du}{dt} \quad \text{.......................(14).}$$

Then, if $v_0 = 0$, $\int_0^t F(t-\tau)\, v_\tau\, d\tau$, where $F(\tau)$ is any arbitrary function which is independent of r and t, and does not become infinite between the limits, will also be a solution of (14); for, substituting in (14), the right-hand side becomes

$$F(0)\, v_t + \int_0^t F'(t-\tau)\, v_\tau\, d\tau = F(t)\, v_0 + \int_0^t F(t-\tau)\frac{dv_\tau}{d\tau}\, d\tau$$

$$= \phi\left(\frac{d}{dr}\right)\int_0^t F(t-\tau)\, v_\tau\, d\tau,$$

if $v_0 = 0$.

507. The second expression on the right-hand side of (13) is the value of $\psi_2 \sin^2\theta$; and it is easily seen that this expression vanishes when $t=0$. Hence it follows that the expression which is obtained from (13) by changing t into τ and V into $F'(t-\tau)\,d\tau$, and integrating the result from t to 0, is also a solution of (1). Now, if $F(0)=0$, it will be found on substituting the above-mentioned expression in (2) and (3) that $F(t)$ is the velocity of the

sphere, supposing it to have started from rest; hence this expression gives the current function due to the motion of a sphere which has started from rest, and which is moving with variable velocity $F(t)$.

In order to obtain the equation of motion of the sphere, we must calculate the resistance due to the liquid; but in doing this we may begin by supposing the velocity to be uniform, and perform the above-mentioned operation at a later stage of the process.

If the impressed force is a constant force, such as gravity, which acts in the direction of motion of the sphere, and Z is the resistance due to the liquid, it follows from (25) of § 490, that

$$Z = 2\pi a \int_0^\pi \left(pa \cos\theta - \rho \frac{d\psi_2}{dt} \sin^2\theta\right)_a \sin\theta \, d\theta,$$

also from (14) of § 486,

$$\frac{dp}{d\theta} = \rho \sin\theta \frac{d^2\psi_1}{dt\,dr} - g\rho a \sin\theta,$$

where ρ is the density of the liquid; also, since

$$\int_0^\pi p \cos\theta \sin\theta \, d\theta = -\tfrac{1}{2}\int_0^\pi \sin^2\theta \frac{dp}{d\theta} d\theta,$$

we obtain

$$Z = -\pi\rho a \frac{d}{dt}\int_0^\pi \left(a\frac{d\psi_1}{dr} + 2\psi_2\right)_a \sin^3\theta \, d\theta + M'g$$

$$= -\frac{M'}{a^2}\frac{d}{dt}\left(a\frac{d\psi_1}{dr} + 2\psi_2\right)_a + M'g,$$

where M' is the mass of the liquid displaced. Now, if V were constant, we should obtain from (13)

$$a\left(\frac{d\psi_1}{dr}\right)_a = -V\{\tfrac{3}{2}\nu t + 3a\sqrt{(\nu t/\pi)} + \tfrac{1}{2}a^2\},$$

and $$(\psi_2)_a = -3Va\{\tfrac{1}{2}\nu t/a + \sqrt{(\nu t/\pi)}\},$$

whence $$\left(a\frac{d\psi_1}{dr} + 2\psi_2\right)_a = -V\{\tfrac{9}{2}\nu t + 9a\sqrt{(\nu t/\pi)} + \tfrac{1}{2}a^2\}.$$

We must now change t into τ, V into $F'(t-\tau)\,d\tau$, and integrate the result with respect to τ from t to 0, and we obtain

$$Z = \frac{M'}{a^2}\frac{d}{dt}\int_0^t F'(t-\tau)\{\tfrac{9}{2}\nu\tau + 9a\sqrt{(\nu\tau/\pi)}\}\,d\tau + \tfrac{1}{2}M'\dot{v} + M'g,$$

and the equation of motion of the sphere is

$$(M+\tfrac{1}{2}M')\dot{v}+\frac{9M'}{a^2}\frac{d}{dt}\int_0^t F'(t-\tau)\{\tfrac{1}{2}\nu\tau+a\sqrt{(\nu\tau/\pi}\}\,d\tau$$
$$=(M-M')g\ldots\ldots\ldots\ldots\ldots(15).$$

Integrating the definite integral by parts, and remembering that $F(0)=0$, the result is

$$\int_0^t F(t-\tau)\{\tfrac{1}{2}\nu+\tfrac{1}{2}a\sqrt{(\nu/\pi\tau)}\}\,d\tau,$$

and, differentiating with respect to t, (15) becomes

$$(M+\tfrac{1}{2}M')\dot{v}+\frac{9M'}{2a^2}\left\{\nu v+a\sqrt{\frac{\nu}{\pi}}\int_0^t\frac{F'(t-\tau)}{\sqrt{\tau}}\,d\tau\right\}$$
$$=(M-M')g\ldots\ldots\ldots\ldots(16).$$

Let σ be the density of the sphere, and let

$$\frac{(\sigma-\rho)g}{\sigma+\frac{1}{2}\rho}=f,\qquad \frac{9\rho}{a^2(2\sigma+\rho)}=k,\qquad \lambda=k\nu\ldots\ldots\ldots(17),$$

then (16) becomes

$$\dot{v}+\lambda v+ka\sqrt{\frac{\nu}{\pi}}\int_0^t\frac{F'(t-\tau)}{\sqrt{\tau}}\,d\tau=f\ldots\ldots\ldots\ldots(18).$$

This is the equation of motion of the sphere, from which $F(t)$ or v must be determined.

508. Up to the present time we have supposed the motion to have commenced from rest, so that $F(0)=0$. Let us now suppose that the sphere was initially projected with velocity V. In order to obtain the equation of motion in this case we may divide the time t, into two intervals h and $t-h$, where h is a very small quantity, which ultimately vanishes. During the first interval let the sphere move from rest under the action of gravity and a very large constant force, which is equal to $(M+\frac{1}{2}M')X$, and then let the large force cease to act. This force must be such as to produce a velocity V at the end of the interval h, whence we must have $V=Xh$, $v=Xt$; and, therefore, $v=Vt/h$. Changing f into $f+X$ in (18), multiplying by $\epsilon^{\lambda t}$, and integrating between the limits t and 0, we obtain

$$v\epsilon^{\lambda t}=-ka\sqrt{\frac{\nu}{\pi}}\int_0^t du\int_0^u \epsilon^{\lambda u}F'(u-\tau)\frac{d\tau}{\sqrt{\tau}}+\int_0^h X\epsilon^{\lambda u}du$$
$$+f\int_0^t\epsilon^{\lambda u}du\ldots\ldots\ldots\ldots\ldots(19).$$

Now $F'(t)$ is composed of two parts: a large part which depends upon X, and which is equal to V/h; and another part which depends upon f, and which we shall continue to denote by $F'(t)$. Hence (19) may be written

$$v\epsilon^{\lambda t} = \frac{X}{\lambda}(\epsilon^{\lambda h} - 1) + \frac{f}{\lambda}(\epsilon^{\lambda t} - 1) - ka\sqrt{\frac{\nu}{\pi}}\int_0^t du \int_0^u F'(u-\tau)\,\epsilon^{\lambda u}\frac{d\tau}{\sqrt{\tau}}$$

$$- ka\sqrt{\frac{\nu}{\pi}}\int_0^t \epsilon^{\lambda u}\chi(u)\,du \quad\text{.......(20)},$$

where
$$\chi(u) = \int_0^u \frac{V d\tau}{h\sqrt{\tau}}.$$

Now $\chi(u)$ depends on X, and therefore vanishes when $u > h$. When $u < h$,

$$\chi(u) = 2Vu^{\frac{1}{2}}/h;$$

therefore

$$\int_0^t \epsilon^{\lambda u}\chi(u)\,du = \int_0^h \frac{2V}{h}\,u^{\frac{1}{2}}\epsilon^{\lambda u}\,du = 0, \text{ when } h = 0.$$

Hence, in the limit when h vanishes, (20) becomes

$$v = V\epsilon^{-\lambda t} + \frac{f}{\lambda}(1 - \epsilon^{-\lambda t})$$

$$- ka\sqrt{\frac{\nu}{\pi}}\int_0^t du \int_0^u \epsilon^{-\lambda(t-u)}F'(u-\tau)\frac{d\tau}{\sqrt{\tau}} \quad\text{.........(21)},$$

and the value of the acceleration is

$$\dot{v} = -V\lambda\epsilon^{-\lambda t} + f\epsilon^{-\lambda t}$$

$$- ka\sqrt{\frac{\nu}{\pi}}\frac{d}{dt}\int_0^t du \int_0^h \epsilon^{-\lambda(t-u)}F'(u-\tau)\frac{d\tau}{\sqrt{\tau}} \quad\text{.............. (22)}.$$

509. It seems almost hopeless to attempt to determine the complete value of F from the preceding equations, but, in the case of many liquids, ν is a small quantity, and (21) and (22) may then be solved by the method of successive approximation. For a first approximation

$$\dot{v} = F'(t) = f\epsilon^{-\lambda t},$$

whence
$$\int_0^t \frac{F'(t-\tau)\,d\tau}{\sqrt{\tau}} = f\int_0^t \frac{\epsilon^{-\lambda\tau}\,d\tau}{\sqrt{(t-\tau)}} \quad\text{..............(23)}.$$

The integral on the right-hand side of (23) cannot be evaluated

in finite terms, and we shall denote it by $\phi(t)$. Putting $\tau = ty$, we obtain

$$\phi(t) = \sqrt{t}\int_0^1 \frac{\epsilon^{-\lambda ty}\,dy}{\sqrt{(1-y)}} \qquad (24),$$

$$= \sqrt{t}\int_0^1 \epsilon^{-\lambda ty}\,\Sigma_0^\infty H_n y^n\,dy,$$

where
$$H_n = \frac{1.3\ldots(2n-1)}{2^n\, n!}.$$

Now
$$\int_0^1 \epsilon^{-\lambda ty}\,dy = \frac{1-\epsilon^{-\lambda t}}{\lambda t}.$$

Therefore
$$\int_0^1 y^n \epsilon^{-\lambda ty}\,dy = (-)^n \left(\frac{d}{d\,.\,\lambda t}\right)^n \frac{1-\epsilon^{-\lambda t}}{\lambda t},$$

and therefore

$$\phi(t) = \sqrt{t}\left\{\frac{1-\epsilon^{-\lambda t}}{\lambda t} + \Sigma_1^\infty (-)^n H_n \left(\frac{d}{d\,.\,\lambda t}\right)^n \frac{1-\epsilon^{-\lambda t}}{\lambda t}\right\} \ldots (25).$$

When t is very large we may replace $(1-\epsilon^{-\lambda t})/\lambda t$ by $(\lambda t)^{-1}$, and we shall obtain

$$\phi(t) = \frac{1}{\lambda\sqrt{t}}\left\{1 + \Sigma_1^\infty \frac{H_n}{(\lambda t)^n}\right\}$$

$$= \frac{1}{\lambda^{\frac{1}{2}}\sqrt{(\lambda t - 1)}},$$

which shows that $\phi(t) \doteq 0$ when $t = \infty$.

Another expression for $\phi(t)$ may be obtained in the form of a series, for

$$\phi(t) = \epsilon^{-\lambda t}\int_0^t \frac{\epsilon^{\lambda\tau}\,d\tau}{\sqrt{\tau}}$$

$$= 2\sqrt{t}\left\{1 - \frac{2\lambda t}{1.3} + \frac{(2\lambda t)^2}{1.3.5} - \cdots \frac{(-)^n(2\lambda t)^n}{1.3\ldots(2n+1)} + \cdots\right\} \qquad (26),$$

by successive integration by parts. The above series is convergent for all values of t, and is zero when $t = \infty$.

For a second approximation, (22) gives

$$\dot{v} = F'(t) = f\epsilon^{-\lambda t} - fka\sqrt{\frac{\nu}{\pi}}\,\frac{d}{dt}\int_0^t \epsilon^{-\lambda u}\,\phi(t-u)\,du \ldots (27),$$

and

$$v = V\epsilon^{-\lambda t} + \frac{f}{\lambda}(1-\epsilon^{-\lambda t}) - fka\sqrt{\frac{\nu}{\pi}}\int_0^t \epsilon^{-\lambda u}\,\phi(t-u)\,du \ldots (28).$$

Let $$\chi(t) = \frac{d}{dt}\int_0^t \epsilon^{-\lambda u}\,\phi(t-u)\,du \qquad (29),$$

and (27) becomes

$$F'(t) = f\epsilon^{-\lambda t} - fka\,(\nu/\pi)^{\frac{1}{2}}\,\chi(t).$$

Whence to a third approximation

$$\dot{v} = -V\lambda\epsilon^{-\lambda t} + f\epsilon^{-\lambda t} - fka\sqrt{\frac{\nu}{\pi}}\,\chi(t)$$
$$+\frac{fk^2a^2\nu}{\pi}\frac{d}{dt}\int_0^t du\int_0^u \epsilon^{-\lambda(t-u)}\,\chi(u-\tau)\,\frac{d\tau}{\sqrt{\tau}}.$$

Let $$\psi(t) = \int_0^t \frac{\chi(\tau)\,d\tau}{\sqrt{(t-\tau)}} \qquad (30),$$

and the last equation becomes

$$\dot{v} = -V\lambda\epsilon^{-\lambda t} + f\epsilon^{-\lambda t} - fka\sqrt{\frac{\nu}{\pi}}\,\chi(t)$$
$$+\frac{fk^2a^2\nu}{\pi}\frac{d}{dt}\int_0^t \epsilon^{-\lambda u}\,\psi(t-u)\,du \qquad (31),$$

and

$$v = \frac{f}{\lambda}(1-\epsilon^{-\lambda t}) + V\epsilon^{-\lambda t} - fka\sqrt{\frac{\nu}{\pi}}\int_0^t \epsilon^{-\lambda u}\phi(t-u)\,du$$
$$+\frac{fk^2a^2\nu}{\pi}\int_0^t \epsilon^{-\lambda u}\,\psi(t-u)\,du \qquad (32).$$

We must now express all the above integrals in terms of $\phi(t)$. From (29) we obtain

$$\chi(t) = \frac{d}{dt}\int_0^t \epsilon^{-\lambda(t-u)}\,\phi(u)\,du$$
$$= \phi(t) - \lambda\int_0^t \epsilon^{-\lambda t+\lambda u}\,\phi(u)\,du$$
$$= \phi(t) - \lambda\epsilon^{-\lambda t}\int_0^t du\int_0^u \epsilon^{\lambda\tau}\,\frac{d\tau}{\sqrt{\tau}}$$

by (24). Changing the order of integration, the last integral

$$= \int_0^t d\tau\int_\tau^t \epsilon^{\lambda\tau}\,\frac{du}{\sqrt{\tau}} = \int_0^t \epsilon^{\lambda\tau}\left(\frac{t}{\sqrt{\tau}} - \sqrt{\tau}\right)d\tau$$
$$= \epsilon^{\lambda t}\left\{\phi(t)\left(t+\frac{1}{2\lambda}\right) - \frac{\sqrt{t}}{\lambda}\right\},$$

whence $$\chi(t) = (\tfrac{1}{2} - \lambda t)\,\phi(t) + \sqrt{t} \qquad (33).$$

Substituting this value of $\chi(t)$ in (30), we obtain

$$\psi(t)=\int_0^t(\tfrac{1}{2}-\lambda\tau)\,\phi(\tau)\,\frac{d\tau}{\sqrt{(t-\tau)}}+\int_0^t\sqrt{\frac{\tau}{t-\tau}}\,d\tau.$$

Now $$\int_0^t\frac{\phi(\tau)\,d\tau}{\sqrt{(t-\tau)}}=\int_0^t d\tau\int_0^\tau\frac{\epsilon^{-\lambda u}\,du}{\sqrt{\{(t-\tau)(\tau-u)\}}}$$

$$=\int_0^t du\int_u^t\frac{\epsilon^{-\lambda u}\,d\tau}{\sqrt{\{(t-\tau)(\tau-u)\}}}$$

$$=\pi\int_0^t\epsilon^{-\lambda u}\,du=\frac{\pi}{\lambda}(1-\epsilon^{-\lambda t})\;\ldots\ldots\ldots(34),$$

also $$\int_0^t\frac{\tau\phi(\tau)d\tau}{\sqrt{(t-\tau)}}=\int_0^t d\tau\int_0^\tau\frac{\tau\epsilon^{-\lambda u}\,du}{\sqrt{\{(t-\tau)(\tau-u)\}}}$$

$$=\int_0^t du\int_u^t\frac{\tau\epsilon^{-\lambda u}\,d\tau}{\sqrt{\{(t-\tau)(\tau-u)\}}}$$

$$=\frac{\pi}{2}\int_0^t(t+u)\,\epsilon^{-\lambda u}\,du$$

$$=\frac{\pi}{2\lambda}\left\{t(1-2\epsilon^{-\lambda t})+\frac{1}{\lambda}(1-\epsilon^{-\lambda t})\right\}\;\ldots.(35),$$

and $$\int_0^t\sqrt{\frac{\tau}{t-\tau}}\,d\tau=\tfrac{1}{2}\pi t\;\ldots\ldots\ldots\ldots\ldots\ldots(36),$$

whence $$\psi(t)=\pi t\epsilon^{-\lambda t}\;\ldots\ldots\ldots\ldots\ldots\ldots(37).$$

Again $$\int_0^t\epsilon^{-\lambda u}\,\psi(t-u)\,du=\pi\epsilon^{-\lambda t}\int_0^t(t-u)\,du$$

$$=\tfrac{1}{2}\pi t^2\epsilon^{-\lambda t}\;\ldots\ldots\ldots\ldots\ldots\ldots(38),$$

whence (31) and (32) finally become

$$\dot{v}=f\epsilon^{-\lambda t}-V\lambda\epsilon^{-\lambda t}-fka\,(\nu/\pi)^{\frac{1}{2}}\{(\tfrac{1}{2}-\lambda t)\,\phi(t)+\sqrt{t}\}$$
$$+fk^2a^2\nu t\epsilon^{-\lambda t}(1-\tfrac{1}{2}\lambda t)\;\ldots\ldots\ldots(39),$$

$$v=\frac{f}{\lambda}(1-\epsilon^{-\lambda t})+V\epsilon^{-\lambda t}-fka\,(\nu/\pi)^{\frac{1}{2}}\left\{\left(t+\frac{1}{2\lambda}\right)\phi(t)-\frac{\sqrt{t}}{\lambda}\right\}$$
$$+\tfrac{1}{2}fk^2a^2\nu t^2\epsilon^{-\lambda t}\;\ldots\ldots\ldots(40).$$

These equations determine to a third approximation the values of the acceleration and velocity of the sphere, when it is projected vertically downwards with velocity V, and allowed to descend under the action of gravity. If the sphere is ascending the sign of g must be reversed.

If no forces are in action we must put $f=0$, and the preceding equations give the values of $\dot{v}$ and v to a first approximation only; but, on referring to (21) and (22), it will be seen that the values of these quantities to a third approximation may be obtained in this case from (39) and (40) by changing f into $-V\lambda$ and expunging the terms $f\epsilon^{-\lambda t}$ and $f\lambda^{-1}(1-\epsilon^{-\lambda t})$. We thus obtain, since $\lambda = k\nu$,

$$\dot{v} = -Vk\nu\epsilon^{-\lambda t} + Vak^2\nu^{\frac{3}{2}}\pi^{-\frac{1}{2}}\{(\tfrac{1}{2}-\lambda t)\phi(t)+\sqrt{t}\}$$
$$-Va^2k^3\nu^2t\epsilon^{-\lambda t}(1-\tfrac{1}{2}\lambda t)\text{............}(41),$$

$$v = V\epsilon^{-\lambda t} + Vak^2\nu^{\frac{3}{2}}\pi^{-\frac{1}{2}}\left\{\left(t+\frac{1}{2\lambda}\right)\phi(t)-\frac{\sqrt{t}}{\lambda}\right\}$$
$$-\tfrac{1}{2}Va^2k^3\nu^2t^2e^{-\lambda t}\text{....................}(42).$$

510. It appears from the preceding equations that the successive terms are multiplied by some power of k as well as of ν. If k is not a very large quantity, and the velocity of the sphere is not very great, the foregoing equations may be expected to give fairly correct results; but if k is a very large quantity, it may happen that, notwithstanding the smallness of ν, $k\nu$ may be so large that some of the terms neglected may be of equal or greater importance than those retained. Now from (17), $k = 9\rho(2\sigma+\rho)^{-1}a^{-2}$; if therefore the sphere is considerably denser than the liquid, k will be small provided a be not very small; but if the sphere be considerably less dense than the liquid, k will approximate towards the limit $9a^{-2}$, and this will be very large if a be small, and $k\nu$ may therefore be large. On the other hand, it should be noticed that when $k\nu$ or λ is large the quantities $\epsilon^{-\lambda t}$ and $\phi(t)$ diminish with great rapidity, and it is therefore by no means impossible that the formulæ may give a fairly accurate representation of the motion even in this case.

All that we can therefore safely infer is this, that in the case of a sphere ascending or descending in a liquid whose kinematic coefficient of viscosity is small compared with the radius of the sphere (all quantities being of course referred to the same units), the formulæ would give approximately correct results, provided the velocity of the sphere were not too great. But, in the case of small bodies descending in a highly viscous liquid, it is possible that the motion represented by the formulæ *may be* very different from the actual motion; and if this should turn out to be the fact, the solution of (18) applicable to this case must be obtained by some different method.

Equation (39) shows that after a very long time has elapsed the acceleration vanishes, and the motion becomes ultimately steady; in other words, the acceleration due to gravity is counterbalanced by the retardation due to the viscosity of the liquid. When this state of things has been reached, the terminal velocity of the sphere is

$$v = \frac{f}{\lambda} = \frac{2a^2}{9\nu}\left(\frac{\sigma}{\rho} - 1\right) g;$$

which agrees with (45) of § 495.

Motion of a Sphere which is rotating about a Fixed Diameter.

511. We shall now consider the motion of a sphere which is surrounded by an infinite liquid, and which is rotating about a fixed diameter.

We shall begin by supposing that the angular velocity of the sphere is uniform and equal to ω, and shall endeavour to obtain an expression for the component velocity of the liquid in a plane perpendicular to the axis of rotation, on the supposition that no slipping takes place at the surface of the sphere.

Assuming that the liquid is initially at rest, it is easily seen that none of the quantities can be functions of ϕ, where r, θ, and ϕ are polar coordinates referred to the centre of the sphere as origin. If, therefore, we neglect squares and products of the velocities, the component velocity W of the liquid, perpendicular to any plane containing the axis of rotation, is determined by the equation

$$\frac{dW}{dt} = \nu\left\{\frac{d^2W}{dr^2} + \frac{2}{r}\frac{dW}{dr} + \frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)W - \frac{W}{r^2\sin^2\theta}\right\},$$

and if in this equation we put $W = w\sin\theta$, where w is a function of r and t only, the equation for w is

$$\frac{d^2w}{dr^2} + \frac{2}{r}\frac{dw}{dr} - \frac{2w}{r^2} = \frac{1}{\nu}\frac{dw}{dt} \quad\text{.................}(43).$$

The value of the tangential stress per unit of area which opposes the motion of the sphere is

$$T = -\nu\rho\left(\frac{1}{r\sin\theta}\frac{dR}{d\phi} + \frac{dW}{dr} - \frac{W}{r}\right),$$

where R is the radial velocity; but, since R is not a function of ϕ,

the value of this stress depends solely on that of W. Now it has been pointed out in the previous Chapter that unless the motion of the sphere is exceedingly slow, the motion of the liquid will not take place in planes perpendicular to the axis of rotation, but the velocity of every particle will have a component in the plane containing the particle and this axis. But since this component does not produce any effect on the motion of the sphere, which it is our object to determine, we may confine our attention solely to the calculation of w.

In addition to (43), w must satisfy the conditions:

(i) At the surface of the sphere $w = a\omega$ for all values of t.

(ii) When $t = 0$, $w = 0$ for all values of r greater than a, the radius of the sphere.

Let $w = R\epsilon^{-\lambda^2 \nu t}$ where R is a function of R alone; substituting in (43), we obtain

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{2R}{r^2} + \lambda^2 R = 0,$$

the solution of which is

$$R = A\,\frac{d}{dr}\left\{\frac{1}{r}\cos\lambda\,(r - a + \alpha)\right\},$$

whence
$$w = A\,\frac{d}{dr}\left\{\frac{\epsilon^{-\lambda^2\nu t}}{r}\cos\lambda\,(r - a + \alpha)\right\}.$$

Integrating this with respect to λ between the limits ∞ and 0, and then changing A into $F(\alpha)$ and integrating the result with respect to α between the same limits, we obtain

$$w = \tfrac{1}{2}\sqrt{\frac{\pi}{\nu t}}\,\frac{d}{dr}\,\frac{1}{r}\int_0^\infty F(\alpha)\exp\left\{-\frac{(r - a + \alpha)^2}{4\nu t}\right\}d\alpha.$$

Performing the differentiation and then integrating by parts, we shall obtain

$$w = -\frac{1}{2r}\sqrt{\frac{\pi}{\nu t}}\int_0^\infty\left\{\frac{F(\alpha)}{r} + F'(\alpha)\right\}\exp\left\{-\frac{(r - a + \alpha)^2}{4\nu t}\right\}d\alpha \ldots (43\text{A}),$$

provided $F(0) = 0$ and $F(\alpha)\,\epsilon^{-\alpha^2} = 0$ when $\alpha = \infty$.

The surface condition (i) will be satisfied if

$$F(\alpha) + aF'(\alpha) = -2a^3\omega/\pi,$$

whence
$$F(\alpha) = -2a^3\omega\pi^{-1}(1 - \epsilon^{-\alpha/a}),$$

the constant of integration being determined so that $F(0) = 0$;

this value of $F(\alpha)$ also satisfies the condition that $F(\alpha)\,\epsilon^{-\alpha^2}=0$ when $\alpha=\infty$. We therefore obtain

$$W=\frac{a^2\omega\sin\theta}{r\sqrt{(\pi\nu t)}}\int_0^\infty\left\{\frac{a}{r}+\left(1-\frac{a}{r}\right)\epsilon^{-\alpha/a}\right\}\exp\left\{-\frac{(r-a+\alpha)^2}{4\nu t}\right\}d\alpha\ldots(44).$$

Putting $r-a+\alpha=2u\sqrt{(\nu t)}$ this becomes

$$W=\frac{2a^2\omega\sin\theta}{r\sqrt{\pi}}\times$$

$$\int_{\frac{r-a}{2\sqrt{(\nu t)}}}^\infty\left\{\frac{a}{r}+\left(1-\frac{a}{r}\right)\exp\left(-\frac{2u\sqrt{(\nu t)}-r+a}{a}\right)\right\}\epsilon^{-u^2}du\ldots\ldots(45).$$

If $r>a$ it follows that $W=0$ when $t=0$. When $r=a$ and $t=0$ the lower limit of the definite integral (45) becomes indeterminate; but since, in this case, we are to have $W=a\omega\sin\theta$, it follows that if we put $k=r-a$ the quantities k and t must vanish in such a manner that when $k=0$ and $t=0$, $k/2\sqrt{(\nu t)}=0$.

When $t=\infty$ we obtain

$$W=\frac{a^3\omega\sin\theta}{r^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46).$$

This equation gives the value of W after a sufficient time has elapsed for the motion to have become steady, and agrees with (47) of § 496.

512. Since the tangential stress per unit of area which opposes the motion of the sphere is

$$T=-\nu\rho a\cdot\frac{d}{dr}\left(\frac{W}{r}\right)_a,$$

the opposing couple is

$$\begin{aligned}G&=-2\pi\nu\rho a^3\int_0^\pi\frac{d}{dr}\left(\frac{W}{r}\right)_a\sin^2\theta\,d\theta\\&=-2\pi\nu\rho a^3\frac{d}{dr}\left(\frac{w}{r}\right)_a\int_0^\pi\sin^3\theta\,d\theta\\&=-\tfrac{8}{3}\pi\nu\rho a^4\frac{d}{dr}\left(\frac{w}{r}\right)_a.\end{aligned}$$

If, therefore, the sphere be acted upon by external forces which produce a couple N', its equation of motion will be

$$\tfrac{8}{15}\sigma a^5\dot\omega+G=N',$$

or

$$\frac{\sigma a\dot\omega}{5\rho}-\nu\frac{d}{dr}\left(\frac{w}{r}\right)_a=N\ldots\ldots\ldots\ldots\ldots\ldots(47),$$

where

$$N=3\rho N'/8a^4.$$

When the motion of the sphere commences from rest the value of w or W cosec θ will be obtained from (45) by changing t into τ, ω into $F'(t-\tau)\,d\tau$, and integrating the result with respect to τ from t to 0, where $F(t)$ is the variable angular velocity of the sphere.

Now, $$\frac{d}{dr}\left(\frac{w}{r}\right)_a = \frac{1}{a}\frac{dw}{dr} - \frac{w}{a^2}.$$

Hence, if ω were uniform we should have

$$\left(\frac{dw}{dr}\right)_a = -2\omega + \frac{2\omega}{\sqrt{\pi}}\int_0^\infty \exp\{-2u\sqrt{(\nu t)}/a - u^2\}\,du - \frac{a\omega}{\sqrt{(\pi\nu t)}}.$$

Putting $u + \sqrt{(\nu t)}/a = \beta$, the definite integral

$$= \epsilon^{\nu t/a^2}\int_{\sqrt{(\nu t)}/a}^\infty \epsilon^{-\beta^2}\,d\beta$$

$$= \epsilon^{\nu t/a^2}\left\{\tfrac{1}{2}\sqrt{\pi} - \frac{\sqrt{(\nu t)}}{a} + \frac{(\nu t)^{\frac{3}{2}}}{3a^3} - \dots\right\}$$

$$= \left(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\nu t}}{a} + \frac{\nu t\sqrt{\pi}}{2a^2} - \dots\right),$$

if νt be small; whence

$$\left(\frac{dw}{dr}\right)_a = -\omega - \frac{2\omega}{a\sqrt{\pi}}\left(\sqrt{\nu t} - \frac{\nu t\sqrt{\pi}}{2a}\right) - \frac{a\omega}{\sqrt{\pi\nu t}}.$$

Changing t into τ, and ω into $F'(t-\tau)\,dt$, (47) becomes

$$\frac{\sigma a\dot{\omega}}{5\rho} + \frac{2\nu\omega}{a} + \frac{2\nu}{a^2\sqrt{\pi}}\int_0^t F'(t-\tau)\left(\sqrt{(\nu\tau)} - \frac{\nu\tau\sqrt{\pi}}{2a}\right)d\tau$$
$$+ \sqrt{\frac{\nu}{\pi}}\int_0^t \frac{F'(t-\tau)}{\sqrt{\tau}}\,d\tau = N \quad\dots\dots\dots\dots\dots(48).$$

Putting $$\frac{10\rho}{\sigma a^2} = k, \quad k\nu = \lambda,$$

(48) becomes

$$\dot{\omega} + \lambda\omega + \frac{k\nu^{\frac{3}{2}}}{a\sqrt{\pi}}\int_0^t F'(t-\tau)\left\{\sqrt{\tau} - \frac{\tau}{2a}\sqrt{(\nu\pi)}\right\}d\tau$$
$$+ \tfrac{1}{2}ka\sqrt{\frac{\nu}{\pi}}\int_0^t F'(t-\tau)\frac{d\tau}{\sqrt{\tau}} = \tfrac{1}{2}kaN \quad\dots\dots\dots\dots\dots(49).$$

Now we have supposed the motion to have commenced from rest under the action of the couple N'; but if the sphere had initially been set in rotation with angular velocity Ω, and then left to itself, it could be shown in the same manner as in § 508 that the equation of motion would be

$$\dot{\omega} + \lambda\omega + \frac{k\nu^{\frac{3}{2}}}{a\sqrt{\pi}}\int_0^t F'(t-\tau)\left\{\sqrt{\tau} - \frac{\tau}{2a}\sqrt{(\nu\pi)}\right\}d\tau$$

$$+ \tfrac{1}{2}ka\sqrt{\frac{\nu}{\pi}}\int_0^t F'(t-\tau)\frac{d\tau}{\sqrt{\tau}} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(50),$$

where $F(0) = \Omega$. Putting $\theta(t)$ for the last two terms, and integrating, we obtain

$$\omega = \Omega\epsilon^{-\lambda t} - \int_0^t \epsilon^{-\lambda(t-u)}\,\theta(u)\,du \quad\ldots\ldots\ldots\ldots(51),$$

$$\dot{\omega} = -\lambda\Omega\epsilon^{-\lambda t} - \frac{d}{dt}\int_0^t \epsilon^{-\lambda(t-u)}\,\theta(u)\,du \quad\ldots\ldots(52).$$

For a first approximation we have

$$\omega = \Omega\epsilon^{-\lambda t}, \quad \dot{\omega} = -\lambda\Omega\epsilon^{-\lambda t} = F'(t).$$

Whence, if ϕ, χ, and ψ have the same meanings as in § 509, a second approximation gives

$$\dot{\omega} = F'(t) = -k\nu\Omega\epsilon^{-\lambda t} + \frac{k^2a\Omega\nu^{\frac{3}{2}}}{2\sqrt{\pi}}\chi(t) \quad\ldots\ldots\ldots(53),$$

$$\omega = \Omega\epsilon^{-\lambda t} + \frac{k^2a\Omega\nu^{\frac{3}{2}}}{2\sqrt{\pi}}\int_0^t \epsilon^{-\lambda(t-u)}\,\phi(u)\,du \quad\ldots\ldots\ldots(54).$$

And a third approximation gives

$$\dot{\omega} = -k\nu\Omega\epsilon^{-\lambda t} + \frac{k^2a\Omega\nu^{\frac{3}{2}}}{2\sqrt{\pi}}\chi(t) + \frac{k^2\nu^{\frac{5}{2}}}{2a\sqrt{\pi}}\frac{d}{dt}\int_0^t du\int_0^u \epsilon^{-\lambda(t-\tau)}\sqrt{\tau}\,d\tau$$

$$-\frac{k^3a^2\nu^2}{4\pi}\frac{d}{dt}\int_0^t \epsilon^{-\lambda(t-u)}\,\psi(u)\,du \quad\ldots\ldots\ldots(55),$$

$$\omega = \Omega\epsilon^{-\lambda t} + \frac{k^2a\Omega\nu^{\frac{3}{2}}}{2\sqrt{\pi}}\int_0^t \epsilon^{-\lambda(t-u)}\,\phi(u)\,du + \frac{k^2\nu^{\frac{5}{2}}}{2a\sqrt{\pi}}\int_0^t du\int_0^u \epsilon^{-\lambda(t-\tau)}\sqrt{\tau}\,d\tau$$

$$-\frac{k^3a^2\nu^2}{4\pi}\int_0^t \epsilon^{-\lambda(t-u)}\,\psi(u)\,du \quad\ldots\ldots\ldots(56).$$

Now we have shown in § 509 that

$$\int_0^t \epsilon^{-\lambda(t-u)}\,\phi(u)\,du = \phi(t)\left(t + \frac{1}{2\lambda}\right) - \frac{\sqrt{t}}{\lambda}.$$

Also $$\int_0^t du\int_0^u \epsilon^{-\lambda(t-\tau)}\sqrt{\tau}\,d\tau = \int_0^t d\tau\int_\tau^t \epsilon^{-\lambda(t-\tau)}\sqrt{\tau}\,du$$

$$= \epsilon^{-\lambda t}\int_0^t \epsilon^{\lambda\tau}\left(t\sqrt{\tau} - \tau^{\frac{3}{2}}\right)d\tau$$

$$= -\frac{1}{2\lambda}\left\{t\phi(t) - \frac{3\sqrt{t}}{\lambda} + \frac{3}{2\lambda}\phi(t)\right\}.$$

And the value of the last integral in (56) is given by (38); whence

$$\omega = \Omega\epsilon^{-\lambda t} + \frac{k^2 a\Omega\nu^{\frac{3}{2}}}{2\sqrt{\pi}}\left\{\phi(t)\left(t+\frac{1}{2\lambda}\right)-\frac{\sqrt{t}}{\lambda}\right\}$$
$$+\frac{k^3\nu^{\frac{5}{3}}}{4a\sqrt{\pi}}\left\{\frac{3\sqrt{t}}{\lambda^2}-\phi(t)\left(\frac{3}{2\lambda^2}+\frac{t}{\lambda}\right)\right\}-\tfrac{1}{8}k^3a^2\nu^2t^2\epsilon^{-\lambda t}\ldots(57),$$

which determines the value of the angular velocity as far as $\nu^{\frac{5}{3}}$.

EXAMPLES.

1. A sphere of radius a is surrounded by viscous liquid which is initially at rest. Prove that if the sphere is constrained to rotate about a fixed diameter with uniform velocity ω, and slipping is supposed to take place at the surface of the sphere, the velocity of the liquid at time t, perpendicular to the plane containing the axis of rotation, is equal to

$$\frac{a^2\omega\sin\theta}{r(\pi\nu t)^{\frac{1}{2}}}\int_0^\infty\left\{\frac{a^2}{r(3k+a)}\left(1+\frac{q\epsilon^{p\alpha}-p\epsilon^{q\alpha}}{p-q}\right)\right.$$
$$\left.+\frac{\epsilon^{p\alpha}-\epsilon^{q\alpha}}{p-q}\right\}\exp\left\{-\frac{(r-a+\alpha)^2}{4\nu t}\right\}d\alpha,$$

where $k=\nu\rho/\beta$; β is the coefficient of sliding friction, and p and q are the roots of the equation

$$ka^2x^2+(3k+a)\,ax+3k+a=0.$$

2. In the last example, if the sphere is *filled* with liquid and *no* slipping is supposed to take place; prove that the velocity of the liquid at time t is equal to

$$\omega r\sin\theta - 2\omega\sin\theta\Sigma\frac{\epsilon^{-\lambda^2\nu t}\sin\lambda a\,S(\lambda r)}{\lambda^2 aS'^2(\lambda a)},$$

where $S(r)$ denotes the spherical function $d(r^{-1}\sin r)/dr$, and the different values of λ are the roots of the equation $S'(\lambda a)=0$.

3. A spherical mass of ice which is surrounded by water is made to rotate with uniform angular velocity ω. After the motion has become steady, the ice is suddenly melted; prove that the

component velocity of the water in the plane perpendicular to the axis of rotation at any subsequent time is

$$\tfrac{1}{2}\omega\,(\pi/\nu t)^{\frac{1}{2}}\sin\theta\,\frac{d}{dr}\,\frac{1}{r}\left(\tfrac{1}{2}\int_0^a \alpha^3 - a^3\int_a^\infty\right)\left[\exp\left\{-\frac{(r-\alpha)^2}{4\nu t}\right\} - \exp\left\{-\frac{(r+\alpha)^2}{4\nu t}\right\}\right]d\alpha.$$

4. A right circular cylinder of radius a is filled with viscous liquid which is initially at rest, and made to rotate with uniform angular velocity ω about its axis. Prove that the velocity of the liquid at time t is equal to

$$2\omega\Sigma\,\frac{\epsilon^{-\lambda^2\nu t}\,J_1(\lambda r)}{\lambda J_1'(\lambda a)} + \omega r,$$

where the different values of λ are the roots of the equation $J_1(\lambda a) = 0$.

5. Prove that if in the last example, the cylinder were surrounded by viscous liquid, the solution of the problem might be obtained from the definite integral

$$\int_0^\infty d\lambda \int_0^a \epsilon^{-\lambda^2\nu t}\,\lambda u\phi(u)\,J_1(\lambda u)\,J_1(\lambda r)\,du,$$

by properly determining $\phi(u)$ so as to satisfy the boundary conditions.

6. A perfectly smooth thin cylindrical shell of radius a, is surrounded by viscous liquid which is at rest, and contains viscous liquid which is rotating as a rigid body with angular velocity ω. By means of the expression for $J_0(\lambda)$ given in Ex. 5, Chapter XII, prove that if the shell be removed, the vorticity at any point of the liquid at any subsequent time is equal to

$$\frac{\omega}{2(\pi\nu t)^{\frac{3}{2}}}\int_0^a du\int_0^\pi d\phi\int_0^\infty u\left[(u\cosh\theta + r\cos\phi) \times \exp\left\{-\frac{(u\cosh\theta + r\cos\phi)^2}{4\nu t}\right\}\right]d\theta.$$

CHAPTER XXIII.

MISCELLANEOUS PROPOSITIONS.

513. THE present Chapter will be devoted to the consideration of certain miscellaneous problems relating to the motion of a viscous liquid.

Steady Motion in Pipes and Canals.

514. Let the pipe be cylindrical and vertical, a its radius, and let us suppose that the liquid has been flowing through the pipe long enough for the motion to have become steady. At a considerable distance from either end of the pipe, the velocity may be regarded as wholly vertical; whence using cylindrical coordinates, $u = v = 0$; and the equation of continuity gives

$$dw/dz = 0 \qquad (1),$$

which shows that w is a function of ϖ alone. Also if the axis of z is measured vertically downwards, we obtain from (23) of § 470,

$$0 = \frac{1}{\rho}\frac{dp}{d\varpi} \qquad (2),$$

$$0 = g - \frac{1}{\rho}\frac{dp}{dz} + \nu\left(\frac{d^2w}{d\varpi^2} + \frac{1}{\varpi}\frac{dw}{d\varpi}\right) \qquad (3).$$

From (2) it follows that p is a function of z alone; hence if we differentiate (3) with respect to z, and take account of (1), we shall obtain

$$\frac{d^2p}{dz^2} = 0,$$

whence

$$p = (A + g\rho)z + \Pi \qquad (4),$$

where Π is the pressure at the origin and A is an undetermined constant. Substituting the value of p from (4) in (3), we obtain

$$\frac{d^2w}{d\varpi^2} + \frac{1}{\varpi}\frac{dw}{d\varpi} - \frac{A}{\mu} = 0,$$

the integral of which is

$$w = \tfrac{1}{4}A\varpi^2/\mu + B\log\varpi + C \quad \text{.................(5).}$$

Since w must not be infinite at the centre of the pipe, $B = 0$. In order to determine C, we must take account of the surface condition

$$\beta w = -T = -\mu\frac{dw}{d\varpi} \quad \text{....................(6),}$$

where β is the coefficient of sliding friction. Substituting in (6) from (5), we obtain

$$C = -\tfrac{1}{2}Aa/\beta - \tfrac{1}{4}Aa^2/\mu,$$

whence

$$w = -\tfrac{1}{4}A\mu^{-1}(a^2 - \varpi^2) - \tfrac{1}{2}Aa/\beta.$$

In order to determine the constant A, we must know the pressure at some other point of the pipe; let Π_1 be the pressure when $z = l$, then from (4)

$$A = (\Pi_1 - \Pi - g\rho l)/l \quad \text{.....................(7).}$$

The flux across any section of the pipe is

$$2\pi\int_0^a w\varpi\, d\varpi = -\tfrac{1}{8}\pi a^4 A/\mu - \tfrac{1}{2}\pi a^3 A/\beta,$$
$$= (\pi a^4/8\mu l + \pi a^3/2\beta l)(\Pi - \Pi_1 + g\rho l).$$

If we suppose the tube to be horizontal and of small cross section, and that the current is maintained by a constant pressure Π at one end, and that there is no slipping, we must put $g = 0$, $\beta = \infty$, and the value flux is

$$\pi a^4(\Pi - \Pi_1)/8\mu l.$$

This result agrees with the result obtained by Poiseuille[1] from his experiments on the flow of liquids through capillary tubes, and furnishes a means of determining the value of μ from experiment.

515. Greenhill has pointed out[2], that the motion of a viscous liquid in a cylindrical pipe of any cross section, when there is no slipping, can be obtained whenever the value of the current function for a frictionless liquid contained in a rotating cylinder

[1] *Mém. des Savants Étrangers*, vol. IX. (1846).
[2] *Proc. Lond. Math. Soc.* vol. XIII. p. 43.

of the same form is known; for when the cross section is not a circle (3) becomes

$$\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + M = 0,$$

where M is a constant. Also $w = 0$ at the surface of the pipe. Now we have shown in § 97, that when frictionless liquid is contained in a rotating cylinder,

$$\frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2} = 0$$

at all points of the liquid; and $\psi = -\frac{1}{2}\omega(x^2 + y^2)$ at the boundary, whence if $\chi = \psi + \frac{1}{2}\omega(x^2 + y^2)$,

$$\frac{d^2\chi}{dx^2} + \frac{d^2\chi}{dy^2} - 2\omega = 0$$

at every point of the liquid, and $\chi = 0$ at the boundary; hence χ satisfies the same conditions as w.

If liquid is flowing in an open channel, and the axis of y be vertical, the conditions to be satisfied at the free surface are

$$y = \text{const.}, \quad dw/dy = 0.$$

If therefore any known value of χ satisfies this condition, we can obtain the corresponding solution for liquid flowing in an open channel.

Motion in Parallel Planes.

516. When the lines of flow of a viscous liquid are parallel straight lines, the determination of the motion depends upon the solution of an equation of the same form as that which determines the motion of heat in two dimensions.

Let the axis of x be parallel to the direction of motion; then $v = 0$, $w = 0$ and the equation of continuity gives $du/dx = 0$, which shows that $u = f(y, z, t)$. If no external forces act, the equations of motion are

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\left(\frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) \text{.................}(8),$$

$$0 = -\frac{1}{\rho}\frac{dp}{dy},$$

$$0 = -\frac{1}{\rho}\frac{dp}{dz}.$$

Hence $d^2p/dx^2 = 0$, and $p = x\phi(t) + \text{const.}$; also since p must not be infinite when $x = \pm\infty$, $\phi(t) = 0$, and (8) becomes

$$\frac{du}{dt} = \nu\left(\frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) \quad\text{.....................(9).}$$

Let us now suppose that the liquid extends to infinity in the positive and negative directions of the axis of z, and that u is a function of y and t only; then (9) becomes

$$\frac{du}{dt} = \nu\frac{d^2u}{dy^2} \quad\text{........................(10).}$$

517. The principal solutions of this equation will now be given.

First, let the liquid be unlimited in the positive and negative directions of the axis of y; and let $u = F(y)$ initially.

A particular solution of (10) is $u = \epsilon^{-\lambda^2\nu t}\cos\lambda y$; and since λ is arbitrary we may integrate this expression with respect to λ between the limits ∞ and $-\infty$, and we thus obtain

$$u = (\pi/\nu t)^{\frac{1}{2}}\,\epsilon^{-y^2/4\nu t}.$$

From the form of (10) it follows that if in this expression we change y into $\beta - y$, the resulting expression will also be a solution, whence multiplying by $F(\beta)/2\pi$ and integrating with respect to β between the limits ∞ and $-\infty$ we obtain

$$u = \tfrac{1}{2}(\pi\nu t)^{-\frac{1}{2}}\int_{-\infty}^{\infty} F(\beta)\,\epsilon^{-(\beta-y)^2/4\nu t}\,d\beta.$$

Putting $\beta - y = 2\alpha\sqrt{\nu t}$, this becomes

$$u = \pi^{-\frac{1}{2}}\int_{-\infty}^{\infty} F(y + 2\alpha\sqrt{\nu t})\,\epsilon^{-\alpha^2}\,d\alpha \quad\text{............(11),}$$

and therefore $u = F(y)$, when $t = 0$. This solution is due to Fourier.

Secondly, let the liquid be bounded by the plane xz; and let its initial velocity be $F(y)$; then

$u = F(y)$, when $t = 0$, provided $y > 0$.

$u = 0$, when $y = 0$, for all values of t.

From the preceding case it follows that a solution of (10) is

$$u = \tfrac{1}{2}(\pi\nu t)^{-\frac{1}{2}}\int_{0}^{\infty} F(\beta)\left\{\epsilon^{-(y-\beta)^2/4\nu t} - \epsilon^{-(y+\beta)^2/4\nu t}\right\}d\beta \quad\text{...(12),}$$

also when $y=0$, u is obviously zero. To find the value of u when $t=0$, we observe that (12) may be written

$$u=\pi^{-1}\int_0^\infty d\lambda\int_0^\infty F(\beta)\,\epsilon^{-\lambda^2\nu t}\{\cos\lambda(y-\beta)-\cos\lambda(y+\beta)\}\,d\beta,$$

$$u=2\pi^{-1}\int_0^\infty d\lambda\int_0^\infty F(\beta)\,\epsilon^{-\lambda^2\nu t}\sin\lambda y\sin\lambda\beta d\beta,$$

which by Fourier's theorem is equal to $F(y)$ when $t=0$, for all values of y between ∞ and 0.

Thirdly, let the liquid be initially at rest, and let the plane xz move with velocity $\phi(t)$. Then

$$u=0 \text{ when } t=0, \text{ and } y>0,$$

$$u=\phi(t) \text{ when } y=0, \text{ for all positive values of } t.$$

Since $(\nu t)^{-\frac{1}{2}}\,\epsilon^{-y^2/4\nu t}$ is a solution of (10), its differential coefficient with respect to y is also a solution; we may therefore put

$$u=\tfrac{1}{2}y\,(\pi\nu/t^3)^{-\frac{1}{2}}\,\epsilon^{-y^2/4\nu t}.$$

Since this expression vanishes when $t=0$, it follows from § 506, that a solution of (10) is

$$u=\tfrac{1}{2}y\,(\pi\nu)^{-\frac{1}{2}}\int_0^t \phi(t-\tau)\,\tau^{-\frac{3}{2}}\,\epsilon^{-y^2/4\nu\tau}\,d\tau.$$

Let $$\tfrac{1}{2}y/(\nu\tau)^{\frac{1}{2}}=\alpha,$$

then $$u=2\pi^{-\frac{1}{2}}\int_{y/2(\nu t)^{\frac{1}{2}}}^\infty \phi(t-y^2/4\nu\alpha^2)\,\epsilon^{-\alpha^2}\,d\alpha \;\dots\dots\dots\dots(13).$$

When $y=0$, $u=\phi(t)$; and when $t=0$, $u=0$, whence (13) is the required solution.

Adding together (12) and (13), the resulting value of u satisfies the following conditions:

$$u=F(y), \text{ when } t=0, \text{ and } y>0,$$

$$u=\phi(t), \text{ when } y=0, \text{ for all positive values of } t.$$

We thus obtain the solution for the motion of a viscous liquid which is initially moving with velocity $F(y)$, and which is bounded by the plane $y=0$ which is moving with velocity $\phi(t)$.

518. By means of the definite integral (13) we can obtain the solution of the following problem.

Let the liquid be divided by the plane $y=0$, which is supposed to be perfectly smooth; and let the liquid on the positive side be set in motion with initial velocity V_1, and let the liquid on the

negative side be set in motion with initial velocity V_2, and then let the plane be removed. It is required to find the velocity at any subsequent time.

In (13) let $\phi(t) = \frac{1}{2}V_2$, and we obtain

$$u_2 = V_2\pi^{-\frac{1}{2}}\int_{y/2(\nu t)^{\frac{1}{2}}}^{\infty} \epsilon^{-a^2}\,da,$$

$$= \tfrac{1}{2}V_2 - V_2\pi^{-\frac{1}{2}}\,\mathrm{Erf}\,y/2(\nu t)^{\frac{1}{2}},$$

adopting the notation $\int_0^x \epsilon^{-u^2}\,du = \mathrm{Erf}\,x$, first introduced by Dr J. W. L. Glaisher.

When $t = 0$ and y is positive, $u_2 = 0$; but when $t = 0$ and y is negative, $u_2 = V_2$.

Again from the form of (10), it is evident that

$$u_1 = V_1\pi^{-\frac{1}{2}}\int_{-y/2(\nu t)^{\frac{1}{2}}}^{\infty} \epsilon^{-a^2}\,da,$$

$$= \tfrac{1}{2}V_1 + V_1\pi^{-\frac{1}{2}}\,\mathrm{Erf}\,y/2(\nu t)^{\frac{1}{2}},$$

is also a solution. When $t = 0$ and y is positive $u_1 = V_1$; but when $t = 0$, and y is negative, $u_1 = 0$. Hence if

$$u = u_1 + u_2,$$

$$= \tfrac{1}{2}(V_1 + V_2) + (V_1 - V_2)\,\pi^{-\frac{1}{2}}\,\mathrm{Erf}\,y/2(\nu t)^{\frac{1}{2}},$$

$$u = V_1, \text{ when } t = 0, \text{ and } y \text{ is positive},$$

$$u = V_2, \text{ when } t = 0, \text{ and } y \text{ is negative}.$$

When t is not zero, the value of u on both sides of the plane $y = 0$ is equal to $\frac{1}{2}(V_1 + V_2)$; *hence the vortex sheet which initially existed instantly disappears.*

From the last three sections it is at once obvious that numerous results furnished by the theory of the Conduction of Heat are capable of a hydrodynamical interpretation and vice versâ.

Waves in a Viscous Liquid.

519. When the motion of a liquid is in two dimensions, and the squares and products of the velocities are neglected, we have shown that the current function satisfies the equation

$$\nabla^2\left(\nabla^2 - \frac{1}{\nu}\frac{d}{dt}\right)\psi = 0 \quad \text{..................(14)},$$

the solution of which is

$$\psi = \psi_1 + \psi_2 \qquad (15),$$

where ψ_1, ψ_2 respectively satisfy the equations

$$\nabla^2 \psi_1 = 0 \qquad (16),$$

$$\nabla^2 \psi_2 - \frac{1}{\nu}\frac{d\psi_2}{dt} = 0 \qquad (17).$$

Also p is determined by the equation

$$\frac{dp}{\rho} = \frac{d}{dt}\left(\frac{d\psi_1}{dx}\,dz - \frac{d\psi_1}{dz}\,dx\right) \qquad (18).$$

In considering the problem of wave motion, it will be convenient as in Chap. XVII. to take the origin in the undisturbed surface, and measure the axis of x in the direction of the propagation of the waves, whilst the axis of z is measured vertically upwards; and we have to find a solution of (14) which represents a train of waves, and which also satisfies the following conditions.

At the free surface the normal and tangential stresses must vanish, whence

$$R = 0, \quad T = 0,$$

or

$$p + 2\mu \frac{d^2\psi}{dx\,dz} = 0 \qquad (19),$$

$$\frac{d^2\psi}{dx^2} - \frac{d^2\psi}{dz^2} = 0 \qquad (20).$$

Also if the liquid is bounded by fixed surfaces, we shall assume that the liquid in contact with such surfaces is at rest.

In order to find a solution of (14) we shall assume that x and t enter in the form of the factor $\epsilon^{\iota mx+kt}$ where $m = 2\pi/\lambda$, λ being the wave length; and the principal object of the investigation is to find the value of k. It is obvious that wave motion will not be possible unless k is a complex quantity whose real part is negative, for this is the only form of k which represents a train of waves whose amplitudes diminish with the time.

520. We shall now investigate the propagation of waves in a liquid which was originally at rest, and whose depth is so great in comparison with the wave length, that it may be regarded as infinite.

Putting

$$\alpha^2 = m^2 + k/\nu \qquad (21),$$

we obtain from (16) and (17)

$$\psi_1 = (A\epsilon^{mz} + B\epsilon^{-mz})\,\epsilon^{\iota mx+kt},$$

$$\psi_2 = (C\epsilon^{\alpha z} + D\epsilon^{-\alpha z})\,\epsilon^{\iota mx+kt}.$$

It will hereafter appear that α is a complex quantity whose real part is positive, therefore since z is measured vertically upwards we must have $B = D = 0$, whence

$$\psi = (A\epsilon^{mz} + C\epsilon^{\alpha z})\,\epsilon^{\iota mx+kt} \quad\ldots\ldots\ldots\ldots\ldots\ldots (22).$$

If η be the elevation

$$\dot{\eta} = -\left(\frac{d\psi}{dx}\right)_{z=0} = -\iota m(A+C)\epsilon^{\iota mx+kt},$$

therefore

$$\eta + \iota mk^{-1}(A + C)\,\epsilon^{\iota mx+kt} = 0 \quad\ldots\ldots\ldots\ldots\ldots (23).$$

Previously to disturbance the pressure $p_1 = -g\rho z$, whence if p' be the increment of the pressure due to the wave motion, we obtain from (18)

$$p' = Ak\rho\iota\epsilon^{mz+\iota mx+kt},$$

and therefore

$$p = -g\rho z + Ak\rho\iota\epsilon^{mz+\iota mx+kt} \quad\ldots\ldots\ldots\ldots\ldots (24).$$

At the free surface $z = \eta$, whence substituting in (19) the values of ψ, η and p from (22), (23) and (24), we obtain

$$(gm/k + k + 2m^2\nu)\,A + (gm/k + 2m\nu\alpha)\,C = 0 \quad\ldots\ldots(25).$$

From (20) we likewise obtain

$$Am^2 + C\alpha^2 + (A + C)\,m^2 = 0,$$

which by (21) becomes

$$2m^2A + (2m^2 + k/\nu)\,C = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(26).$$

Eliminating A and C between (25) and (26) we obtain

$$k^2 + 4m^2k\nu + gm + 4m^4\nu^2 - 4m^3\nu^2\alpha = 0 \quad\ldots\ldots\ldots (27),$$

which by virtue of (21) is a biquadratic equation for determining k.

When ν is small $\alpha = (k/\nu)^{\frac{1}{2}}$ approximately, and therefore the last two terms of (27) may be neglected; we thus obtain

$$k^2 + 4km^2\nu + mg = 0,$$

the solution of which is

$$k = -2m^2\nu \pm \sqrt{(4m^4\nu^2 - gm)},$$

$$= -2m^2\nu \pm \iota n \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28)$$

approximately, where $n^2 = gm$.

Taking the lower sign we obtain from (26)

$$C = -2Am^2\nu\iota/n,$$

and from (21) $\qquad \alpha^2 = -m^2 - \iota n/\nu.$

On account of the smallness of ν, the first term may be neglected, whence if

$$\beta = (n/2\nu)^{\frac{1}{2}},$$

$$\alpha = (1-\iota)\beta,$$

and therefore

$$\psi = A\epsilon^{-2m^2\nu t}\{\epsilon^{mz} - 2m^2\nu n^{-1}\iota\epsilon^{(1-\iota)\beta z}\}\,\epsilon^{\iota mx - \iota nt}.$$

Rejecting the imaginary part, we obtain

$$\psi = A\epsilon^{-2m^2\nu t}\{\epsilon^{mz}\cos(mx-nt) + 2m^2\nu n^{-1}\epsilon^{\beta z}\sin(mx-\beta z-nt)\}.$$

On account of the smallness of ν the last term is insensible; also if V be the velocity of propagation, and λ the wave length,

$$m = 2\pi/\lambda, \quad (mg)^{\frac{1}{2}} = n = 2\pi V/\lambda, \quad \text{whence } V^2 = g\lambda/2\pi, \text{ and}$$

$$\psi = A\epsilon^{-8\pi^2\nu t/\lambda^2 + mz}\cos\frac{2\pi}{\lambda}(x - Vt),$$

and therefore the modulus of decay is $\lambda^2/8\pi^2\nu$,

The preceding value of ψ represents a train of waves whose amplitude diminishes with the time; it also appears that the diminution due to viscosity is very much less in the case of long waves than in the case of short ones.

If we were to proceed to a second approximation, it would be found that

$$k = -2m^2\nu\{1 - m(2\nu/n)^{\frac{1}{2}}\} - \iota n\{1 - m^3(2\nu^3/n^3)^{\frac{1}{2}}\}.$$

521. Let us now suppose that ν is large. Putting $x^2 = k/\nu$, (27) becomes

$$x^4 + 4m^2x^2 + gm/\nu^2 + 4m^4 - 4m^3(m^2+x^2)^{\frac{1}{2}} = 0.$$

In this put $x = m\tan\theta$, and we obtain

$$\tan^4\theta + 4\tan^2\theta + g/m^3\nu^2 = 4(\sec\theta - 1).$$

On account of the largeness of ν, the term $g/m^3\nu^2$ is very small, and may therefore be neglected, whence dividing out by $\sec\theta - 1$ and putting $\sec\theta = y$, we obtain

$$y^3 + y^2 + 3y - 1 = 0.$$

This cubic has one real root which is approximately equal to ·29, whence

$$k/\nu = m^2\tan^2\theta = -m^2 \times \cdot 92$$

approximately. The other two roots lead to complex values of k/ν whose real parts are approximately equal to $-m^2 \times 3{\cdot}3$.

The factor $\sec\theta - 1$ leads to $k = 0$, which corresponds to no motion, and this root must therefore be rejected. We therefore see that the real part of k is a large negative quantity, and therefore the motion rapidly dies away.

522. These results are entirely in accordance with what is observed in the case of viscous liquids. If for example a jet of air were directed for a short time to the surface of a slightly viscous liquid such as water, waves would be observed to diverge from the point of application of the jet, whose amplitudes gradually diminish as the time advances, until the motion ultimately dies away. But if the jet were applied to the surface of a highly viscous liquid such as treacle or glycerine, waves would not be excited. The immediate effect of the jet of air would be to produce a depression in the neighbourhood of its point of application, and as soon as it had ceased, the liquid would sluggishly move so as to fill up the depression, and would very soon come to rest.

523. We shall now solve the same problem when the depth of the liquid is finite and equal to h. In this case we shall have

$$\psi = (A\cosh mz + B\sinh mz + C\cosh\alpha z + D\sinh\alpha z)\,\epsilon^{\iota mx+kt}.$$

The conditions to be satisfied at the bottom of the liquid are that

$$d\psi/dz = 0, \quad d\psi/dx = 0 \text{ when } z = -h.$$

Putting

$L = \cosh mh$, $M = \sinh mh$, $P = \cosh\alpha h$, $Q = \sinh\alpha h$, these conditions give

$$AL - BM + CP - DQ = 0 \;\ldots\ldots\ldots\ldots(29),$$

$$(AM - BL)\,m + (CQ - DP)\,\alpha = 0 \;\ldots\ldots\ldots\ldots(30).$$

Also $$\dot\eta = -\left(\frac{d\psi}{dx}\right)_{z=0} = -\,\iota m\,(A + C)\,\epsilon^{\iota mx+kt},$$

whence $$\eta + \iota mk^{-1}(A + C)\,\epsilon^{\iota mx+kt} = 0 \;\ldots\ldots\ldots\ldots\ldots(31).$$

From (18) we obtain

$$p = -\,g\rho z + k\rho\iota\,(A\sinh mz + B\cosh mz)\,\epsilon^{\iota mx+kt} \;\ldots\ldots(32).$$

At the free surface $z = \eta$, whence substituting the values of ψ, η and p in (19) we obtain

$$Agm/k + B\,(2m^2\nu + k) + Cgm/k + 2Dm\alpha\nu = 0 \;\ldots\ldots(33).$$

Also from (20) and (21)

$$2Am^2\nu + (2m^2\nu + k)\, C = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (34).$$

Eliminating A, B, C, D from (29), (30), (33) and (34) we obtain,

$$\begin{vmatrix} 2m^2\nu, & L, & Mm, & gm/k, \\ 0, & -M, & -Lm, & 2m^2\nu + k, \\ 2m^2\nu + k, & P, & Q\alpha, & gm/k, \\ 0, & -Q, & -P\alpha, & 2mn\alpha\nu, \end{vmatrix} = 0 \ldots (35).$$

This is the equation for determining k. For the purpose of obtaining an approximate solution which is applicable to waves in water, we shall neglect the square of ν; and the determinantal equation then becomes

$$\begin{aligned} &-2m^2\nu\,(PM\alpha gm/k + QLm^2 g/k + k\alpha) \\ &+(2m^2\nu + k)\,[-Qm\,\{M(2m^2\nu + k) + Lgm/k\} + L\alpha\,\{P(2m^2\nu + k) - 2m^2\nu\} \\ &\qquad + Mm\alpha\,(Pg/k + 2Mm\nu)] = 0. \end{aligned}$$

Since ν is small α is large, and therefore P and Q are large, we may therefore put $P/Q = 1$. Dividing out by Q, it will be found that the largest terms are those which are multiplied by α; whence retaining the most important terms only, it will be found that the equation for k reduces to

$$k^2 + 4km^2\nu + mg \tanh mh = 0,$$

the solution of which is

$$\begin{aligned} k &= -2m^2\nu \pm \surd(4m^4\nu^2 - mg \tanh mh), \\ &= -2m^2\nu \pm \iota\,(mg \tanh mh)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots\ldots (36). \end{aligned}$$

Hence the velocity of propagation is determined by the equation

$$V^2 = (g\lambda/2\pi) \tanh (2\pi h/\lambda),$$

which agrees with the result found in § 384. The modulus of decay is $\lambda^2/8\pi^2\nu$ as before.

If the depth of the liquid is small compared with the wave length, we may replace tan mh by mh, and (36) becomes approximately

$$= -2m^2\nu \pm m\iota\,(gh)^{\frac{1}{2}},$$

which shows that long waves travel with a velocity of propagation approximately equal to $(gh)^{\frac{1}{2}}$, and that the amplitude diminishes with the time[1].

[1] In connection with this subject, a paper by Lord Rayleigh "On the Circulation of Air observed in Kundt's Tubes," *Phil. Trans.* 1884, may be consulted.

Instability of Viscous Liquids.

524. The instability of viscous liquids has been studied experimentally by Prof. Osborne Reynolds[1]. His experiments consisted in causing water to flow from a large cistern through a tube 4 feet 6 inches long, and by means of suitable appliances a fine stream of coloured liquid was made to pass down the centre of the tube along with the water. The results of experiments made with three different tubes, whose respective diameters were 1 inch, $\frac{1}{2}$ inch and $\frac{1}{4}$ inch were as follows. When the velocity was sufficiently small, the streak of coloured liquid extended in a straight line through the tube, and if the liquid in the cistern were slightly disturbed, the streak would oscillate in the tube about its mean position, but showed no tendency to mix with the water. It thus appeared that for small velocities the motion was stable.

As the velocity was gradually increased, it was found that as soon as it had attained a certain critical value, the coloured liquid commenced to mix with the water, and the motion became unstable; but the point at which instability commenced was always at a considerable distance from the extremity of the tube at which the water flowed in, and the intervening portion was perfectly clear.

Any increase in the velocity caused the point at which instability commenced, to approach this extremity, but Reynolds did not succeed in obtaining a velocity large enough to make the region of stability altogether disappear.

On examining the unstable portion of the liquid by the light of an electric spark, the mass of colour was found to consist of a number of distinct curls, showing the existence of eddies.

When the water was kept at constant temperature, and the cistern as still as possible, it was found that the critical velocity was inversely proportional to the diameter of the tube; and also that if the viscosity of the water was diminished by increasing its temperature, the critical velocity diminished directly as the coefficient of viscosity.

It was also found that the critical velocity was very sensitive to disturbance of the water before entering the tube; and it was

[1] "On the Motion of Water and the Law of Resistance in Parallel Channels," *Phil. Trans.* 1883, p. 935.

only by the greatest care as to uniformity of the temperature of the cistern and the stillness of the water, that consistent results could be obtained. This showed that the steady motion was unstable for large disturbances long before the critical velocity was reached,—a fact which agreed with the full-blown manner in which eddies appeared.

If t be the temperature expressed in degrees centigrade, D the diameter of the tube, and if

$$P = (1 + \cdot 0336t + \cdot 00221t^2)^{-1} \propto \nu,$$

Reynolds found that the critical velocity was given by the formula

$$U = P/BD, \text{ where } B = 43\cdot 79;$$

and he concluded, that the probable condition of stability of a viscous liquid is that Uc/ν should be less than a certain numerical quantity, where c is a length U and a velocity, which define the linear scale and scale of velocity of the system, and ν the kinematic coefficient of viscosity.

For the mathematical treatment of this question, the following papers by Sir W. Thomson may be consulted[1].

On the Oscillations of a Viscous Spheroid.

525. The oscillations of a viscous spheroid have been investigated by Prof. G. H. Darwin[2] and Prof. Lamb[3]; we shall now proceed to give the investigation of the latter.

The equations

$$(\nabla^2 + k^2)\,u = 0,\ (\nabla^2 + k^2)\,v = 0,\ (\nabla^2 + k^2)\,w = 0 \ldots\ldots (37),$$

subject to the condition

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (38),$$

are of frequent occurrence in a variety of physical investigations, and we shall commence by obtaining the solution of these equations subject to the condition of finiteness at the origin.

Let $u = \psi_n \phi_n$ where ϕ_n is a solid harmonic of degree n, and

[1] "On the Stability of Fluid Motion," *Phil. Mag.* (5) vol. XXIV. pp. 188 and 272. "On the Propagation of Laminar Motion through a turbulently moving Inviscid Liquid," *Ibid.* p. 342.

[2] "On the Bodily Tides of Viscous Spheroids," *Phil. Trans.* 1879.

[3] *Proc. Lond. Math. Soc.* vol. XIII. p. 51.

ψ_n is a function of r alone; substituting in (37), taking account of the value of ∇^2 in (15) of § 10, and remembering that

$$d\phi_n/dr = n\phi_n/r,$$

we shall find that the equation to be satisfied by ψ_n is

$$\frac{d^2\psi_n}{dr^2} + \frac{2(n+1)}{r}\frac{d\psi_n}{dr} + k^2\psi_n = 0 \quad\ldots\ldots\ldots\ldots(39).$$

If we put $\psi_n = R_n/r^n$, the equation for R_n is

$$\frac{d^2R_n}{dr^2} + \frac{2}{r}\frac{dR_n}{dr} - \frac{n(n+1)}{r}R_n + k^2R_n = 0 \quad\ldots\ldots\ldots(40).$$

The properties of the function R_n have been fully discussed by the authorities cited below[1]; for our present purpose it will only be necessary to consider that solution of (39) which is finite when $r = 0$.

Integrating (39) in a series of ascending powers of r we obtain

$$\psi_n = 1 - \frac{k^2r^2}{2\,.\,2n+3} + \frac{k^4r^4}{2\,.\,4\,.\,2n+3\,.\,2n+5} - \ldots\ldots(41),$$

from which it is evident that

$$\psi_0 = (kr)^{-1}\sin kr.$$

By means of (41) we can easily prove that

$$r\frac{d\psi_{n-1}}{dr} = -\frac{k^2r^2}{2n+1}\psi_n \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(42),$$

$$\frac{r}{2n+1}\frac{d\psi_n}{dr} = \psi_{n-1} - \psi_n \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(43),$$

$$k^2r^2\psi_{n+1} = (2n+1)(2n+3)(\psi_n - \psi_{n-1}) \quad\ldots\ldots\ldots(44).$$

It follows from (42) that

$$\psi_n = (-)^n\,1\,.\,3\ldots\ldots(2n+1)\left(\frac{1}{z}\frac{d}{dz}\right)^n\frac{\sin z}{z} \quad\ldots\ldots\ldots(45),$$

where $z = kr$.

Let ϕ_n, χ_n be any two spherical solid harmonics of degree n;

[1] Stokes, "On the Communication of Vibrations from a Vibrating Body to a surrounding Gas," *Phil. Trans.* 1868.

C. Niven, "On the Conduction of Heat in Ellipsoids of Revolution," *Phil. Trans.* 1880.

C. Niven, "On the Induction of Electric Currents in Infinite Plates and Spherical Shells," *Phil. Trans.* 1881.

Lamb, "On Electrical Motions in a Spherical Conductor," *Phil. Trans.* 1883.

Lord Rayleigh, *Theory of Sound*, vol. II, ch. XVII.

then $d\phi_{n+1}/dx$ is obviously a solid harmonic of degree n; also by substituting each of the functions

$$y\frac{d\chi_n}{dz} - z\frac{d\chi_n}{dy}, \quad z\frac{d\chi_n}{dx} - x\frac{d\chi_n}{dz}, \quad x\frac{d\chi_n}{dy} - y\frac{d\chi_n}{dx}$$

in Laplace's equation, it can be shown that the latter three functions are also solid harmonics of degree n. It therefore follows that

$$u = \psi_n\left(\frac{d\phi_{n+1}}{dx} + y\frac{d\chi_n}{dz} - z\frac{d\chi_n}{dy}\right) \text{.............} (46)$$

with symmetrical expressions for v and w are respectively solutions of (37). These expressions do not however satisfy (38), for taking account of (42) and remembering that

$$x\frac{d\phi_n}{dx} + y\frac{d\phi_n}{dy} + z\frac{d\phi_n}{dz} = n\phi_n,$$

we find that

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = -\frac{(n+1)k^2}{2n+3}\psi_{n+1}\phi_{n+1} \text{..........} (47).$$

The function $r^{2n+5}d(r^{-2n-3}\phi_{n+1})/dx$ is a homogeneous function of degree $n+2$; substituting in Laplace's equation, and using the equation

$$x\phi_n = \frac{r^2}{2n+1}\left(\frac{d\phi_n}{dx} - r^{2n+1}\frac{d}{dx}\frac{\phi_n}{r^{2n+1}}\right) \text{...........} (48)$$

which can be immediately verified, it can be shown that this function is a solid harmonic of degree $n+2$. We may therefore assume

$$u' = Ak^2\psi_{n+2}r^{2n+5}\frac{d}{dx}\frac{\phi_{n+1}}{r^{2n+3}} \text{................} (49),$$

where A is a constant, with symmetrical expressions for v' and w'. Substituting in (38) we obtain

$$\begin{aligned}\frac{du'}{dx} + \frac{dv'}{dy} + \frac{dw'}{dz} &= -Ak^2r(n+2)\phi_{n+1}\frac{d\psi_{n+2}}{dr} \\ &\quad - Ak^2(n+2)(2n+5)\phi_{n+1}\psi_{n+2} \\ &= -Ak^2(n+2)(2n+5)\phi_{n+1}\psi_{n+1}\end{aligned}$$

by (43). Hence if

$$A = -\frac{(n+1)}{(n+2)(2n+3)(2n+5)},$$

it follows from (48) that the complete solution of (37) and (38) is $u+u'$ &c. Whence we may put

$$u = \Sigma\left\{\psi_n\left(\frac{d\phi_{n+1}}{dx} + y\frac{d\chi_n}{dz} - z\frac{d\chi_n}{dy}\right) - \frac{(n+1)k^2\psi_{n+2}r^{2n+5}}{(n+2)(2n+3)(2n+5)}\frac{d}{dx}\frac{\phi_{n+1}}{r^{2n+3}}\right\}$$
$$\text{...............} (50).$$

526. It is important to notice that the solution of the proposed system of equations consists of two distinct types, which may be thus written:

1st type

$$u = \psi_n \left(y \frac{d\chi_n}{dz} - z \frac{d\chi_n}{dy} \right) \dots\dots\dots\dots\dots\dots (51).$$

2nd type

$$u = \psi_{n-1} \frac{d\phi_n}{dx} - \frac{nk^2 r^2 \psi_{n+1}}{(n+1)(2n+1)(2n+3)} r^{2n+1} \frac{d}{dx} \frac{\phi_n}{r^{2n+1}} \dots (52).$$

It should also be noticed that the solutions of the first type make

$$xu + yv + zw = 0 \dots\dots\dots\dots\dots\dots\dots\dots (53),$$

whilst those of the second type make

$$xu + yv + zw = n\psi_n \phi_n \dots\dots\dots\dots\dots\dots (54).$$

527. We shall now apply the preceding results to determine the small oscillations of a nearly spherical mass of viscous liquid.

Since the motion is small, we may neglect the squares and products of the velocities, and the equations of motion are

$$\frac{du}{dt} = \nu \nabla^2 u + \frac{dQ}{dx}, \text{ \&c., \&c.} \dots\dots\dots\dots\dots\dots (55),$$

where $Q = -p/\rho + V$ and V is the attraction potential; also if we assume that the time enters in the form of the factor $\epsilon^{-\alpha t}$, these may be written

$$(\nabla^2 + k^2)\, u = -\nu^{-1} dQ/dx \dots\dots\dots\dots\dots\dots (56),$$

where $k^2 = \alpha/\nu$, and the exponential factor is omitted.

From (56) combined with (38) it follows that $\nabla^2 Q = 0$, and therefore

$$u = -\alpha^{-1} dQ/dx + \text{terms of types (51) and (52).}$$

The condition to be satisfied at the free surface is that the stress must be zero, hence the boundary conditions are

$$\left.\begin{aligned} xP + yU + zT &= 0 \\ xU + yQ + zS &= 0 \\ xT + yS + zR &= 0 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots (57).$$

Substituting the values of P, Q ...... from § 468 (19), these become

$$2xu_x + y(u_y + v_x) + z(u_z + w_x) = px/\mu, \text{ \&c., \&c.,}$$

or

$$\left(r\frac{d}{dr} - 1 \right) u + \frac{d}{dx}(xu + yv + zw) = \frac{px}{\mu} \text{ \&c., \&c.} \dots\dots (58).$$

528. From (53) it follows that the terms of the first type represent motions which are everywhere perpendicular to the radius vector, and which are therefore unaffected by gravitation. Hence the vibrations represented by the two types are independent of one another.

As regards the vibrations of the first type, (56) and (57) are satisfied by

$$Q=0, \quad u=\psi_n\left(y\frac{d\chi_n}{dz}-z\frac{d\chi_n}{dy}\right),$$

also since there is no radial motion, the surface value of V is constant. Substituting in (58), we obtain after reduction

$$\left[a\frac{d\psi_n}{dr}+(n-1)\psi_n\right]=0,$$

where the square brackets denote surface values. This equation determines the values of k which are all real, and hence the values of α.

529. Since Q can be expanded in a series of spherical solid harmonics of the form ΣQ_n, the vibrations of the second type can be expressed by equations of the form

$$u=-\frac{1}{a}\frac{dQ_n}{dx}+\psi_{n-1}\frac{d}{dx}\left(\frac{r^n}{a^n}T_n\right)-\frac{n}{n+1}(\psi_n-\psi_{n-1})\left(\frac{r}{a}\right)^{2n+1}\frac{d}{dx}\left(\frac{a^{n+1}}{r^{n+1}}T_n\right);$$

where T_n is a spherical surface harmonic of degree n, and a is the mean radius of the sphere. Substituting in (58) and taking account of (48), we obtain at the surface

$$\left(r\frac{d}{dr}-1\right)u=-\frac{n-2}{a}\frac{dQ_n}{dx}+\frac{d}{dx}\left(\frac{r^n}{a^n}T_n\right)\cdot\left(r\frac{d}{dr}+n-2\right)\psi_{n-1}$$
$$-\frac{n}{n+1}\frac{d}{dx}\left(\frac{a^{n+1}}{r^{n+1}}T_n\right)\cdot\left(r\frac{d}{dr}+n-2\right)(\psi_n-\psi_{n-1})\text{......}(59).$$

Also at the surface,

$$\frac{d}{dx}(xu+yv+zw)=\frac{d}{dx}\left(-\frac{nQ_n}{a}+\frac{nr^n}{a^n}\psi_nT_n\right)$$
$$=-\frac{n}{a}\frac{dQ_n}{dx}+\frac{xT_n}{a}\frac{d\psi_n}{dr}+n\psi_n\frac{d}{dx}\left(\frac{r^n}{a^n}T_n\right)$$
$$=-\frac{n}{a}\frac{dQ_n}{dx}+n\psi_{n-1}\frac{d}{dx}\left(\frac{r^nT_n}{a^n}\right)$$
$$+n(\psi_n-\psi_{n-1})\frac{d}{dx}\left(\frac{a^{n+1}T_n}{r^{n+1}}\right)\text{......}(60).$$

Let the equation of the free surface be

$$r = a + S_n,$$

where S_n is another spherical surface harmonic, then by § 371, the value of the potential at an internal point is,

$$V = \text{const.} - \tfrac{2}{3}\pi\rho r^2 + \frac{4\pi a\rho S_n}{2n+1}\left(\frac{r}{a}\right)^n,$$

and therefore the value of V at the surface is

$$V = \text{const.} - \frac{8\pi a\rho\,(n-1)\,S_n}{3\,(2n+1)}.$$

Putting $$\beta^2 = \frac{2n\,(n-1)\,g}{(2n+1)\,a} \quad\text{......................(61)},$$

remembering that $g = \tfrac{4}{3}\pi\rho a$, and suitably choosing the const., the value of V may be written

$$V = -\,n^{-1}a\beta^2 S_n.$$

Hence

$$\begin{aligned} px/\mu &= -\,(Q_n - V)\,x/\nu \\ &= -\,(Q_n + n^{-1}a\beta^2 S_n)\,x/\nu \quad\text{..........................(62)} \\ &= -\,\frac{a^2}{\nu\,(2n+1)}\left\{\frac{dQ_n}{dx} - a^{2n+1}\frac{d}{dx}\frac{Q_n}{r^{2n+1}}\right\} \\ &\quad -\frac{a^3\beta^2}{\nu\,(2n+1)}\left\{\frac{d}{dx}\frac{r^nS_n}{a^n} - a^{n+1}\frac{d}{dx}\frac{S_n}{r^{n+1}}\right\} \quad\text{..........(63)}, \end{aligned}$$

by (48). But at the surface we have also the kinematical condition

$$\begin{aligned} dS_n/dt &= -\,\alpha S_n = (xu + yv + zw)/a \\ &= -\,(nQ_n/\alpha - n\psi_n T_n)/a. \end{aligned}$$

Accordingly from (62) and (63) we obtain

$$\begin{aligned} p/\mu = -\,\frac{k^2a^2}{(2n+1)\,\alpha}\Bigg\{&\left(1 + \frac{\beta^2}{\alpha^2}\right)\left(\frac{dQ_n}{dx} - a^{2n+1}\frac{d}{dx}\frac{Q_n}{r^{2n+1}}\right) \\ &- \frac{\beta^2}{\alpha}\left(\frac{d}{dx}\frac{r^nT_n}{a^n} - \frac{d}{dx}\frac{a^{n+1}T_n}{r^{n+1}}\right)\psi_n\Bigg\}. \end{aligned}$$

Collecting our results, and substituting in the three equations (58), and equating separately the surface harmonics of degrees $n-1$ and $n+1$, we shall obtain after reduction

$$\begin{aligned} &(2n-2)\,Q_n - [rd\psi_{n-1}/dr + (2n-2)\,\psi_{n-1}]\,\alpha T_n \\ &\qquad = k^2a^2\,[(1 + \beta^2/\alpha^2)\,Q_n - (\beta/\alpha)^2\,\psi_n\alpha T_n]/(2n+1) \text{......(64)}, \end{aligned}$$

and

$$\begin{aligned} &(2n+1)\,n\,[(rd/dr - 3)\,(\psi_n - \psi_{n-1})\,\alpha T_n] \\ &\qquad = -\,(n+1)\,k^2a^2\,[(1 + \beta^2/\alpha^2)\,Q_n - (\beta/\alpha)^2\,\psi_n\alpha T_n] \text{......(65)}, \end{aligned}$$

where the square brackets indicate surface values. These equations may be written

$$\{2(n-1)(2n+1)-k^2a^2(1+\beta^2/\alpha^2)\}[Q_n]$$
$$=[2(n-1)(2n+1)\psi_{n-1}-k^2a^2(1+\beta^2/\alpha^2)\psi_n]\,\alpha T_n,$$

and

$$(n+1)k^2a^2(1+\beta^2/\alpha^2)[Q_n]+[-(n+1)k^2a^2(1+\beta^2/\alpha^2)\psi_n$$
$$+(2n+4)na\,d\psi_n/dr+(2n+1)k^2a^2\psi_n]\,\alpha T_n=0.$$

Eliminating Q_n and T_n, we shall finally obtain

$$\frac{(2n-2)k^2a^2}{(2n+1)^2(2n+3)}\left(1+\frac{\beta^2}{\alpha^2}\right)\psi_{n+1}=\left\{2n-2-\frac{k^2a^2}{2n+1}\left(1+\frac{\beta^2}{\alpha^2}\right)\right\}$$
$$\times\left\{\frac{\psi_n}{n+1}-\frac{(2n+4)n\psi_{n+1}}{(n+1)(2n+1)(2n+3)}\right\}\ldots\ldots\ldots\ldots(66).$$

530. This is the equation for determining the values of ka; it can be approximately solved either when the viscosity is very large or very small.

When ν is large k and α are small, and $\psi_n=1$; putting

$$\zeta=k^2a^2\beta^2/(2n+1)\alpha^2=\beta^2a^2/(2n+1)\nu\alpha\ldots\ldots\ldots(67),$$

(66) becomes

$$\frac{2(n-1)\zeta}{(2n+1)(2n+3)}=\frac{2(n-1)-\zeta}{n+1}\left\{1+\frac{(2n+4)n}{(2n+1)(2n+3)}\right\},$$

approximately. Solving for ζ and substituting in (67), we obtain

$$\alpha=\frac{nga\nu^{-1}}{2(n+1)^2+1},$$

a result which was first obtained by Prof. G. H. Darwin[1].

On the other hand when ν is small, it is evident from § 446 that α is nearly equal to $\iota\beta$, so that k is large. From (45) it is easily seen that the most important part of ψ_n is

$$(-)^n\,1\,.\,3\,.\,5\ldots\ldots(2n+1)(ka)^{-n-1}\sin(ka+\tfrac{1}{2}n\pi).$$

It thus appears that the ratio ψ_{n+1}/ψ_n is of the order $(ka)^{-1}$ and (66) becomes approximately

$$2(n-1)(2n+1)-k^2a^2(1+\beta^2/\alpha^2)=0.$$

This leads to

$$\alpha/\iota\beta=1+(n-1)(2n+1)/k^2a^2$$
$$=1+(n-1)(2n+1)\nu/\iota\beta a^2,$$

[1] *Phil. Trans.* 1879, p. 10.

whence, $$\alpha = \iota\beta + (n-1)(2n+1)\nu a^{-2},$$

whence the modulus of decay is

$$\tau = a^2/(n-1)(2n+1)\nu.$$

From this result it appears that the oscillations of a globe of moderate dimensions are very slightly affected by such an amount of viscosity as is ordinarily met with in nature.

For a globe of the same size as the earth, and of the same kinematic viscosity as water, we have on the C. G. S. system $a = 6{\cdot}37 \times 10^8$, $\nu = {\cdot}014$; and Prof. Lamb finds that the value of τ for the oscillation of longest period, i.e. $n = 2$, is

$$\tau = 1{\cdot}84 \times 10^{11} \text{ years}.$$

Prof. Darwin has found that the viscosity of pitch near the freezing temperature is $\mu = 1{\cdot}3 \times 10^8 \times g$, hence taking $g = 980$, we find

$$\tau = 150 \text{ hours}.$$

This is the modulus of decay of the slowest oscillation of a globe of the size of the earth, having the density of water and the viscosity of pitch.

The oscillations of a cylindrical mass of rotating viscous liquid have been discussed by Mr G. H. Bryan, in a paper which is to be published in the *Proc. Camb. Phil. Soc.* vol. VI.

EXAMPLES.

1. A current of liquid is made to flow through an infinitely long rectangular tube one of whose sides is smooth, and the other is rough; after the motion has become steady the forces which maintain the motion cease to act; prove that the velocity at distance y from the smooth side, at time t after the forces have ceased to act, is

$$u = 8U\pi^{-2}\Sigma_1^{\infty}(2n+1)^{-2}\exp\{-(2n+1)^2\pi^2\nu t/4a^2\}\cos(2n+1)\pi y/2a,$$

where a is the width of the tube, and U is the velocity of the liquid in contact with the smooth side in steady motion.

2. An unlimited mass of viscous liquid is divided by the plane $y=0$. The liquid on the positive side of the plane is at rest, whilst the liquid on the negative side is initially moving with a velocity parallel to x, which is equal to $U\epsilon^{\beta x}\cos\beta x$, where $\beta=(2\nu)^{-\frac{1}{2}}$ and x is negative. Prove that if the plane be removed, the velocity of the liquid at any subsequent time is

$$\frac{U}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{(\nu t)}}}^{\infty} \cos\left(t-\frac{x^2}{4\nu u^2}\right)\epsilon^{-u^2}\,du.$$

3. A viscous liquid is contained between two smooth parallel plane boundaries $x=\pm a$, unlimited in one direction, and closed by a rough plane $y=0$, which is capable of movement in its own plane in the direction of the axis of x. Prove that if the rough plane be moved with constant velocity V, so small that the squares and products of the velocity of the fluid may be neglected, then after the motion has become steady, the current function is given by the equation

$$\psi=-2Vy\pi^{-1}\tan^{-1}\{\cos(\pi x/2a)\operatorname{cosec}(\pi y/2a)\}.$$

4. A viscous liquid occupies the space between two infinite parallel planes. One of the planes is fixed, whilst the other moves parallel to itself with a simple harmonic motion $A\cos nt$. Show that the tangential force on the fixed plane has a maximum value per unit of area, which is equal to

$$\frac{2A\lambda\mu n}{(\cosh 2\lambda l-\cos 2\lambda l)^{\frac{1}{2}}},$$

where l is the distance between the planes, $\lambda^2=\rho n/2\mu$, and the fluid in contact with the plane is assumed to adhere to it.

5. Prove that when viscous liquid is flowing steadily through a cylindrical tube of any section, the curves of equal velocity are the same as the relative stream lines of a frictionless liquid filling an equal cylinder, due to any plane motion of the cylinder perpendicular to its generating lines, the viscous liquid being supposed to adhere to the sides of the tube.

If the section of the tube be the ellipse $(y/b)^2+(z/c)^2=1$, prove that the velocity of the liquid at any point is

$$Ab^2c^2(1-y^2/b^2-z^2/c^2)/(b^2-c^2),$$

and that the molecular rotation is

$$Ab^2c^2(y^2/b^4+z^2/c^4)^{\frac{1}{2}}/(b^2+c^2).$$

6. A stream of liquid free from viscosity, is flowing with velocity u along a straight smooth pipe of length l and uniform circular section of radius a. Supposing the pipe suddenly to become rough, the coefficient of sliding friction having a given value β, and the fluid viscous, the coefficient of viscosity having a given value μ, find the additional pressure which must be applied at the end from which the stream is flowing in order to keep the efflux unaltered.

7. A mass of air bounded by two infinite planes perpendicular to the axis of y and distant y_1 apart is in motion, the motion being the same in all planes parallel to xy. Form equations to determine the motion taking account of internal friction, and show that if it be periodic in x and t, and the direct effect of friction be limited to a thin layer near the planes, then neglecting terms involving the squares and higher powers of the velocities, a solution is given by

$$u = \cos kx \left[\epsilon^{-\beta(y+y_1)} \cos \{nt - \beta(y+y_1)\} - \cos nt\right],$$

$$v = -(k/\beta\sqrt{2}) \sin kx$$
$$\times \left[yy_1^{-1} \cos(nt - \tfrac{1}{4}\pi) + \epsilon^{-\beta(y+y_1)} \cos\{nt - \tfrac{1}{4}\pi - \beta(y+y_1)\}\right],$$

where $\beta = (n/2\nu)^{\frac{1}{2}}$; ν is the kinematic coefficient of viscosity; $p = a^2\rho$ and

$$k = \pm na^{-1}\{1 + \tfrac{1}{2}(1-\iota) y_1^{-1}(4n/\nu)^{-\frac{1}{2}}\}.$$

8. A viscous fluid flows between two parallel planes and the motion is slightly disturbed; prove that if u be the velocity at right angles to the planes which is supposed to vary as $\epsilon^{\iota pt + \iota mz}$, where t is the time and z the axis parallel to the plane, then u satisfies the equation

$$\left\{\left(\frac{d^2}{dx^2} - m^2\right)\left(\frac{d^2}{dx^2} - m^2 - \frac{\iota p}{\nu}\right) + \frac{V\iota m}{2a^2\nu}\right\} u = 0,$$

where ν is the kinematic coefficient of viscosity, $2a$ the distance between the planes, V the original velocity of the fluid midway between the planes, the axis of x being perpendicular to the planes.

APPENDIX.

I. A CLASS of functions closely allied to toroidal functions has been recently investigated by Mr Hobson[1]. These functions are spherical harmonics of complex degree $-\frac{1}{2}+n\iota$; and appear to have been first studied by Mehler[2], by whom they were called *Kegelfunctionen*, which may be translated *Conical Harmonics*. They are also discussed by Heine[3], and there is a short note upon them by Mr Burnside[4].

Mr Hobson has applied these functions to the solution of a variety of problems in Electricity and the Conduction of Heat. He has also obtained the current function due to the motion parallel to its axis of a spindle-shaped solid, formed by the revolution of a segment of a circle round its chord. The result is expressed in the form of a definite integral, which although elegant from an analytical point of view, is of the same complicated character as the corresponding result in the case of the cardioid which is given in § 271.

II. The investigation of § 332—3 is not quite satisfactory in the case of a hollow vortex; for in deducing the value of β_1 we have employed the value of ψ', whereas in this case ψ' does not exist, and the value of β_1 must be deduced from that of ψ.

In this case the boundary condition is

$$-\frac{d\psi}{dk}b\beta_1 \sin\xi + \frac{d\psi}{d\xi} = 0 \qquad (1).$$

Now the most important terms of $d\psi/dk$ are of zero order, and therefore the first term of (1) is of the second order, therefore to the first order, the condition becomes

$$\frac{d\psi}{d\xi} = 0 \qquad (2).$$

[1] "On a Class of Spherical Harmonics of Complex Degree," *Trans. Camb. Phil. Soc.* vol. XIV. p. 211.

[2] Ueber eine mit den Kugel- und Cylinderfunctionen verwandte Function, *Elbing* 1870; and *Crelle*, vol. LXVIII.

[3] *Kugelfunctionen*, vol. II. p. 217.

[4] *Mess. Math.* vol. XIV. p. 122.

Now from (67) of § 329,

$$\frac{d\psi}{d\xi} = -2Va^2k\sin\xi + k\sin\xi\{A_0R_0 + A_1R_1(b/k)\cos\xi\}$$
$$-(1-k\cos\xi)A_1R_1(b/k)\sin\xi,$$

whence retaining terms of the first order only, (2) becomes

$$2Va^2b + \tfrac{1}{2}bA_0(L-2) + \tfrac{1}{2}A_1 = 0,$$

whence by (69, a), $\beta_1 = 0$; which shows that β_1 is of the second order of small quantities.

III. In a paper which is to be published in the *American Journal of Mathematics*, I have employed toroidal functions to investigate the steady motion of an annular mass of liquid, whose cross section is small compared with its aperture, and which is rotating like a rigid body about its axis of unequal moment.

If the cross section of the ring is given by the equation

$$k = b(1 + \beta_1\cos\xi + \beta_2\cos 2\xi + \ldots\ldots),$$

it is shown by a process similar to that employed in Chapter XIV., that the values of the β's in terms of b and the angular velocity ω, can be obtained to any degree of approximation that may be desired; and the value of β_1 to a first approximation is

$$\beta_1 = \tfrac{1}{12}b(31 + 12\lambda - 8\log 4/b) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $\lambda = \omega^2/4\pi\rho$.

It is also assumed that no hollow space exists within the liquid, and this leads to the following inequality which expresses the condition that the pressure should not become negative inside the ring, viz.,

$$\lambda^2 - (\tfrac{2}{3}\log 4/b - \tfrac{19}{12})b^2\lambda + b^2 > 0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If therefore the radius of the critical circle be taken as the unit of length, we may assign to b and λ any values which make β_1 a small quantity, and which also satisfy (2).

If $b = \cdot 1$, then $\qquad \beta_1 = \cdot 0124 + 10\lambda,$

and if we put $\lambda = \cdot 01$, the left-hand side of (2) is equal to $\cdot 9224$, and is therefore positive, and therefore $\beta_1 = \cdot 1124$. Hence

$$b = \cdot 1, \quad \beta_1 = \cdot 1124, \quad \omega^2/4\pi\rho = \cdot 01,$$

are solutions of the problem.

IV. Equations (12) of § 467 may be proved in a somewhat different manner as follows.

We have shown in § 464 that there are three planes mutually at right angles to one another, across which the tangential stresses are zero. Let e', f', g' be the rates of extension perpendicular to these